

ON ϕ - n -ABSORBING PRIMARY IDEALS OF COMMUTATIVE RINGS

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ABSTRACT. All rings are commutative with $1 \neq 0$ and n is a positive integer. Let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function where $\mathfrak{J}(R)$ denotes the set of all ideals of R . We say that a proper ideal I of R is ϕ - n -absorbing primary if whenever $a_1, a_2, \dots, a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$, either $a_1 a_2 \cdots a_n \in I$ or the product of a_{n+1} with $(n-1)$ of a_1, \dots, a_n is in \sqrt{I} . The aim of this paper is to investigate the concept of ϕ - n -absorbing primary ideals.

1. INTRODUCTION

Throughout this paper R will be a commutative ring with a nonzero identity. In [4], Anderson and Smith called a proper ideal I of a commutative ring R to be *weakly prime* if whenever $a, b \in R$ and $0 \neq ab \in I$, either $a \in I$ or $b \in I$. In [9], Bhatwadekar and Sharma defined a proper ideal I of an integral domain R to be *almost prime* (resp. *m -almost prime*) if for $a, b \in R$ with $ab \in I \setminus I^2$, (resp. $ab \in I \setminus I^m, m \geq 3$) either $a \in I$ or $b \in I$. This definition can obviously be made for any commutative ring R . Later, Anderson and Batanieh [3] gave a generalization of prime ideals which covers all the above mentioned definitions. Let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. A proper ideal I of R is said to be ϕ -*prime* if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, $a \in I$ or $b \in I$. Since $I \setminus \phi(I) = I \setminus (I \cap \phi(I))$, without loss of generality we may assume that $\phi(I) \subseteq I$. We henceforth make this assumption. Weakly primary ideals were first introduced and studied by Ebrahimi Atani and Farzalipour in [10]. A proper ideal I of R is called *weakly primary* if for $a, b \in R$ with $0 \neq ab \in I$, either $a \in I$ or $b \in \sqrt{I}$. In [22], Yousefian Darani called a proper ideal I of R to be ϕ -*primary* if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, then either $a \in I$ or $b \in \sqrt{I}$. He defined the map $\phi_\alpha : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ as follows:

- (1) $\phi_\emptyset : \phi(I) = \emptyset$ defines primary ideals.
- (2) $\phi_0 : \phi(I) = 0$ defines weakly primary ideals.
- (3) $\phi_2 : \phi(I) = I^2$ defines almost primary ideals.
- (4) $\phi_m (m \geq 2) : \phi(I) = I^m$ defines m -almost primary ideals.
- (5) $\phi_\omega : \phi(I) = \bigcap_{m=1}^{\infty} I^m$ defines ω -primary ideals.
- (6) $\phi_1 : \phi(I) = I$ defines any ideals.

Given two functions $\psi_1, \psi_2 : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for each $J \in \mathfrak{J}(R)$. Note in this case that

$$\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{m+1} \leq \phi_m \leq \cdots \leq \phi_2 \leq \phi_1.$$

Badawi in [5] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal I of R to be a *2-absorbing ideal* of R if whenever

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$a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [2] generalized the concept of 2-absorbing ideals to n -absorbing ideals. According to their definition, a proper ideal I of R is called an n -absorbing (resp. *strongly n -absorbing*) ideal if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \dots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals I_1, \dots, I_{n+1} of R), then there are n of the a_i 's (resp. n of the I_i 's) whose product is in I . Thus a strongly 1-absorbing ideal is just a prime ideal. Clearly a strongly n -absorbing ideal of R is also an n -absorbing ideal of R . Anderson and Badawi conjectured that these two concepts are equivalent, e.g., they proved that an ideal I of a Prüfer domain R is strongly n -absorbing if and only if I is an n -absorbing ideal of R , [2, Corollary 6.9]. They also gave several results relating strongly n -absorbing ideals. The concept 2-absorbing ideals has another generalization, called weakly 2-absorbing ideals, which has studied in [8]. A proper ideal I of R is a *weakly 2-absorbing ideal* of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Generally, Mostafanasab and Yousefian Darani [15] called a proper ideal I of R to be a *weakly n -absorbing* (resp. *strongly weakly n -absorbing*) ideal if whenever $0 \neq a_1 \cdots a_{n+1} \in I$ for $a_1, \dots, a_{n+1} \in R$ (resp. $0 \neq I_1 \cdots I_{n+1} \subseteq I$ for ideals I_1, \dots, I_{n+1} of R), then there are n of the a_i 's (resp. n of the I_i 's) whose product is in I . Clearly a strongly weakly n -absorbing ideal of R is also a weakly n -absorbing ideal of R . Let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. We say that a proper ideal I of R is a ϕ - n -absorbing ideal of R if $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for $a_1, a_2, \dots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I \setminus \phi(I)$ for ideals I_1, \dots, I_{n+1} of R) implies that there are n of the a_i 's (resp. n of the I_i 's) whose product is in I . Notice that ϕ - n -absorbing ideals of a commutative ring R have already been investigated by Ebrahimpour and Nekooei [11] as $(n, n+1)$ - ϕ -prime ideals.

We call a proper ideal I of R to be a ϕ - n -absorbing primary (resp. *strongly ϕ - n -absorbing primary*) ideal of R if $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some elements $a_1, a_2, \dots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ and $I_1 \cdots I_{n+1} \not\subseteq \phi(I)$ for ideals I_1, \dots, I_{n+1} of R) implies that either $a_1 a_2 \cdots a_n \in I$ or the product of a_{n+1} with $(n-1)$ of a_1, a_2, \dots, a_n is in \sqrt{I} (resp. either $I_1 I_2 \cdots I_n \subseteq I$ or the product of I_{n+1} with $(n-1)$ of I_1, I_2, \dots, I_n is in \sqrt{I}).

We can define the map $\phi_\alpha : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ as follows:

Let I be a ϕ_α - n -absorbing primary ideal of R . Then

- (1) $\phi_\emptyset(I) = \emptyset \Rightarrow I$ is an n -absorbing primary ideal.
- (2) $\phi_0(I) = 0 \Rightarrow I$ is a weakly n -absorbing primary ideal.
- (3) $\phi_2(I) = I^2 \Rightarrow I$ is an almost n -absorbing primary ideal.
- (4) $\phi_m(I) = I^m$ ($m \geq 2$) $\Rightarrow I$ is an m -almost n -absorbing primary ideal.
- (5) $\phi_\omega(I) = \bigcap_{m=1}^\infty I^m \Rightarrow I$ is an ω - n -absorbing primary ideal.
- (6) $\phi_1(I) = I \Rightarrow I$ is any ideal.

Some of our results use the $R(+)M$ construction. Let R be a ring and M be an R -module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$.

In [19], Quatararo et al. said that a commutative ring R is a u -ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals. They show that every Bézout ring is a u -ring. Moreover, they proved that every Prüfer domain is a u -domain.

Let R be a ring and $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. In section 2, we give some basic properties of ϕ - n -absorbing primary ideals. For instance, we prove

that if ϕ reverses the inclusion and for every $1 \leq i \leq k$, I_i is a ϕ - n_i -absorbing primary ideal of R such that $\sqrt{I_i}$ is a ϕ - n_i -absorbing ideal of R , respectively, then $I_1 \cap I_2 \cap \cdots \cap I_k$ and $I_1 I_2 \cdots I_k$ are two ϕ - n -absorbing primary ideals of R where $n = n_1 + n_2 + \cdots + n_k$. It is shown that a Noetherian domain R is a Dedekind domain if and only if a nonzero n -absorbing primary ideal of R is in the form of $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals M_1, M_2, \dots, M_i of R and some positive integers t_1, t_2, \dots, t_i . Moreover, we prove that if I is an ideal of a ring R such that $\sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_n$ where M_i 's are maximal ideals of R , then I is an n -absorbing primary ideal of R . We show that if I is an ϕ - n -absorbing primary ideal of R that is not an n -absorbing primary ideal, then $I^{n+1} \subseteq \phi(I)$.

In section 3, we investigate ϕ - n -absorbing primary ideals of direct products of commutative rings. For example, it is shown that if R is an indecomposable ring and J is a finitely generated ϕ - n -absorbing primary ideal of R , where $\phi \leq \phi_{n+2}$, then J is weakly n -absorbing primary. Let $n \geq 2$ be a natural number and $R = R_1 \times \cdots \times R_{n+1}$ be a decomposable ring with identity. Then we prove that R is a von Neumann regular ring if and only if every proper ideal of R is an n -almost n -absorbing primary ideal of R if and only if every proper ideal of R is an ω - n -absorbing primary ideal of R .

In Section 4, we study the stability of ϕ - n -absorbing primary ideals with respect to idealization. As a result of this section, we establish that if I is a proper ideal of R and M is an R -module such that $IM = M$, then $I(+)M$ is an n -almost n -absorbing primary (resp. ω - n -absorbing primary) ideal of $R(+)M$ if and only if I is an n -almost n -absorbing primary (resp. ω - n -absorbing primary) ideal of R .

In section 5, we prove that over a u -ring R the two concepts of strongly ϕ - n -absorbing primary ideals and of ϕ - n -absorbing primary ideals are coincide. Moreover, if R is a Prüfer domain and I is an ideal of R , then I is an n -absorbing primary (resp. a weakly n -absorbing primary) ideal of R if and only if $I[X]$ is an n -absorbing primary (resp. a weakly n -absorbing primary) ideal of $R[X]$.

2. PROPERTIES OF ϕ - n -ABSORBING PRIMARY IDEALS

Let n be a positive integer. Consider elements a_1, \dots, a_n and ideals I_1, \dots, I_n of a ring R . Throughout this paper we use the following notations:

$a_1 \cdots \widehat{a_i} \cdots a_n$: i -th term is excluded from $a_1 \cdots a_n$.

Similarly; $I_1 \cdots \widehat{I_i} \cdots I_n$: i -th term is excluded from $I_1 \cdots I_n$.

It is obvious that any n -absorbing primary ideal of a ring R is a ϕ - n -absorbing primary ideal of R . Also it is evident that the zero ideal is a weakly n -absorbing primary ideal of R . Assume that p_1, p_2, \dots, p_{n+1} are distinct prime numbers. We know that the zero ideal $I = \{0\}$ is a weakly n -absorbing primary ideal of the ring $\mathbb{Z}_{p_1 p_2 \cdots p_{n+1}}$. Notice that $p_1 p_2 \cdots p_{n+1} = 0 \in I$, but neither $p_1 p_2 \cdots p_n \in I$ nor $p_1 \cdots \widehat{p_i} \cdots p_{n+1} \in \sqrt{I} = \text{Nil}(\mathbb{Z}_{p_1 p_2 \cdots p_{n+1}})$ for every $1 \leq i \leq n$. Hence I is not an n -absorbing primary ideal of $\mathbb{Z}_{p_1 p_2 \cdots p_{n+1}}$.

Remark 2.1. Let I be an ideal of a ring R and $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function.

- (1) I is ϕ -primary if and only if I is ϕ -1-absorbing primary.
- (2) If I is ϕ - n -absorbing primary, then it is ϕ - i -absorbing primary for all $i > n$.

- (3) If I is ϕ -primary, then it is ϕ - n -absorbing primary for all $n > 1$.
- (4) If I is ϕ - n -absorbing primary for some $n \geq 1$, then there exists the least $n_0 \geq 1$ such that I is ϕ - n_0 -absorbing primary. In this case, I is ϕ - n -absorbing primary for all $n \geq n_0$ and it is not ϕ - i -absorbing primary for $n_0 > i > 0$.

Remark 2.2. If I is a radical ideal of a ring R , then clearly I is a ϕ - n -absorbing primary (resp. strongly ϕ - n -absorbing primary) ideal if and only if I is a ϕ - n -absorbing (resp. strongly ϕ - n -absorbing) ideal.

Theorem 2.3. Let R be a ring and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then the following conditions are equivalent:

- (1) I is ϕ - n -absorbing primary;
- (2) For every elements $x_1, \dots, x_n \in R$ with $x_1 \cdots x_n \notin \sqrt{I}$,

$$(I :_R x_1 \cdots x_n) \subseteq [\cup_{i=1}^{n-1} (\sqrt{I} :_R x_1 \cdots \widehat{x}_i \cdots x_n)] \cup (I :_R x_1 \cdots x_{n-1}) \cup (\phi(I) :_R x_1 \cdots x_n).$$

Proof. (1) \Rightarrow (2) Suppose that $x_1, \dots, x_n \in R$ such that $x_1 \cdots x_n \notin \sqrt{I}$. Let $a \in (I :_R x_1 \cdots x_n)$. So $ax_1 \cdots x_n \in I$. If $ax_1 \cdots x_n \in \phi(I)$, then $a \in (\phi(I) :_R x_1 \cdots x_n)$. Assume that $ax_1 \cdots x_n \notin \phi(I)$. Since $x_1 \cdots x_n \notin \sqrt{I}$, then either $ax_1 \cdots x_{n-1} \in I$, i.e., $a \in (I :_R x_1 \cdots x_{n-1})$ or for some $1 \leq i \leq n-1$ we have $ax_1 \cdots \widehat{x}_i \cdots x_n \in \sqrt{I}$, i.e., $a \in (\sqrt{I} :_R x_1 \cdots \widehat{x}_i \cdots x_n)$ or $x_1 x_2 \cdots x_n \in \sqrt{I}$. By assumption the last case is not hold. Consequently

$$(I :_R x_1 \cdots x_n) \subseteq [\cup_{i=1}^{n-1} (\sqrt{I} :_R x_1 \cdots \widehat{x}_i \cdots x_n)] \cup (I :_R x_1 \cdots x_{n-1}) \cup (\phi(I) :_R x_1 \cdots x_n).$$

(2) \Rightarrow (1) Let $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some $a_1, a_2, \dots, a_{n+1} \in R$ such that $a_1 a_2 \cdots a_n \notin I$. Then $a_1 \in (I :_R a_2 \cdots a_{n+1})$. If $a_2 \cdots a_{n+1} \in \sqrt{I}$, then we are done. Hence we may assume that $a_2 \cdots a_{n+1} \notin \sqrt{I}$ and so by part (2),

$$(I :_R a_2 \cdots a_{n+1}) \subseteq [\cup_{i=2}^n (\sqrt{I} :_R a_2 \cdots \widehat{a}_i \cdots a_{n+1})] \cup (I :_R a_2 \cdots a_n) \cup (\phi(I) :_R a_2 \cdots a_{n+1}).$$

Since $a_1 a_2 \cdots a_{n+1} \notin \phi(I)$ and $a_1 a_2 \cdots a_n \notin I$, the only possibility is that $a_1 \in \cup_{i=2}^n (\sqrt{I} :_R a_2 \cdots \widehat{a}_i \cdots a_{n+1})$. Then $a_1 a_2 \cdots \widehat{a}_i \cdots a_{n+1} \in \sqrt{I}$ for some $2 \leq i \leq n$. Consequently I is ϕ - n -absorbing primary. \square

Let R be an integral domain with quotient field K . Badawi and Houston [6] defined a proper ideal I of R to be strongly primary if, whenever $ab \in I$ with $a, b \in K$, we have $a \in I$ or $b \in \sqrt{I}$. In [22], a proper ideal I of R is called strongly ϕ -primary if whenever $ab \in I \setminus \phi(I)$ with $a, b \in K$, we have either $a \in I$ or $b \in \sqrt{I}$. We say that a proper ideal I of R is *quotient ϕ - n -absorbing primary* if whenever $x_1 x_2 \cdots x_{n+1} \in I$ with $x_1, x_2, \dots, x_{n+1} \in K$, we have either $x_1 x_2 \cdots x_n \in I$ or $x_1 \cdots \widehat{x}_i \cdots x_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$.

Proposition 2.4. Let V be a valuation domain with the quotient field K , and let $\phi : \mathfrak{J}(V) \rightarrow \mathfrak{J}(V) \cup \{\emptyset\}$ be a function. Then every ϕ - n -absorbing primary ideal of V is quotient ϕ - n -absorbing primary.

Proof. Assume that I is a ϕ - n -absorbing primary ideal of V . Let $x_1x_2\cdots x_{n+1} \in I$ for some $x_1, x_2, \dots, x_{n+1} \in K$ such that $x_1x_2\cdots x_n \notin I$. If $x_{n+1} \notin V$, then $x_{n+1}^{-1} \in V$, since V is valuation. So $x_1\cdots x_nx_{n+1}x_{n+1}^{-1} = x_1\cdots x_n \in I$, a contradiction. Hence $x_{n+1} \in V$. If $x_i \in V$ for every $1 \leq i \leq n$, then there is nothing to prove. If $x_i \notin V$ for some $1 \leq i \leq n$, then $x_1\cdots \widehat{x_i}\cdots x_{n+1} \in I \subseteq \sqrt{I}$. Consequently, I is quotient ϕ - n -absorbing primary. \square

Proposition 2.5. *Let R be a von Neumann regular ring and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then I is a ϕ - n -absorbing primary ideal of R if and only if $e_1e_2\cdots e_{n+1} \in I \setminus \phi(I)$ for some idempotent elements $e_1, e_2, \dots, e_{n+1} \in R$ implies that either $e_1e_2\cdots e_n \in I$ or $e_1\cdots \widehat{e_i}\cdots e_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$.*

Proof. Notice the fact that any finitely generated ideal of a von Neumann regular ring R is generated by an idempotent element. \square

Theorem 2.6. *Let R be a ring and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. If I is a ϕ - n -absorbing primary ideal of R such that $\sqrt{\phi(I)} = \phi(\sqrt{I})$, then \sqrt{I} is a ϕ - n -absorbing ideal of R .*

Proof. Let $x_1x_2\cdots x_{n+1} \in \sqrt{I} \setminus \phi(\sqrt{I})$ for some $x_1, x_2, \dots, x_{n+1} \in R$ such that $x_1\cdots \widehat{x_i}\cdots x_{n+1} \notin \sqrt{I}$ for every $1 \leq i \leq n$. Then there is a natural number m such that $x_1^mx_2^m\cdots x_{n+1}^m \in I$. If $x_1^mx_2^m\cdots x_{n+1}^m \in \phi(I)$, then $x_1x_2\cdots x_{n+1} \in \sqrt{\phi(I)} = \phi(\sqrt{I})$, which is a contradiction. Since I is ϕ - n -absorbing primary, our hypothesis implies that $x_1^mx_2^m\cdots x_n^m \in I$. Hence $x_1x_2\cdots x_n \in \sqrt{I}$. Therefore \sqrt{I} is a ϕ - n -absorbing ideal of R . \square

Corollary 2.7. *Let I be an n -absorbing primary ideal of R . Then $\sqrt{I} = P_1 \cap P_2 \cap \cdots \cap P_i$ where $1 \leq i \leq n$ and P_i 's are the only distinct prime ideals of R that are minimal over I .*

Proof. In Theorem 2.6, suppose that $\phi = \phi_\emptyset$. Now apply [2, Theorem 2.5]. \square

Theorem 2.8. *Let R be a ring, and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function that reverses the inclusion. Suppose that for every $1 \leq i \leq k$, I_i is a ϕ - n_i -absorbing primary ideal of R such that $\sqrt{I_i} = P_i$ is a ϕ - n_i -absorbing ideal of R , respectively. Set $n := n_1 + n_2 + \cdots + n_k$. The following conditions hold:*

- (1) $I_1 \cap I_2 \cap \cdots \cap I_k$ is a ϕ - n -absorbing primary ideal of R .
- (2) $I_1I_2\cdots I_k$ is a ϕ - n -absorbing primary ideal of R .

Proof. Set $L = I_1 \cap I_2 \cap \cdots \cap I_k$. Then $\sqrt{L} = P_1 \cap P_2 \cap \cdots \cap P_k$. Suppose that $a_1a_2\cdots a_{n+1} \in L \setminus \phi(L)$ for some $a_1, a_2, \dots, a_{n+1} \in R$ and $a_1\cdots \widehat{a_i}\cdots a_{n+1} \notin \sqrt{L}$ for every $1 \leq i \leq n$. By, $\sqrt{L} = P_1 \cap P_2 \cap \cdots \cap P_k$ is ϕ - n -absorbing, then $a_1a_2\cdots a_n \in P_1 \cap P_2 \cap \cdots \cap P_k$. We claim that $a_1a_2\cdots a_n \in L$. For every $1 \leq i \leq k$, P_i is ϕ - n_i -absorbing and $a_1a_2\cdots a_n \in P_i \setminus \phi(P_i)$, then there exist elements $1 \leq \beta_1^i, \beta_2^i, \dots, \beta_{n_i}^i \leq n$ such that $a_{\beta_1^i}a_{\beta_2^i}\cdots a_{\beta_{n_i}^i} \in P_i$. If $\beta_r^l = \beta_s^m$ for two pairs l, r and m, s , then

$$a_{\beta_1^1}a_{\beta_2^1}\cdots a_{\beta_{n_1}^1}\cdots a_{\beta_1^l}a_{\beta_2^l}\cdots a_{\beta_r^l}\cdots a_{\beta_{n_l}^l}\cdots$$

$$a_{\beta_1^m}a_{\beta_2^m}\cdots \widehat{a_{\beta_s^m}}\cdots a_{\beta_{n_m}^m}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k} \in \sqrt{L}.$$

Therefore $a_1 \cdots \widehat{a_{\beta_s^m}} \cdots a_n a_{n+1} \in \sqrt{L}$, a contradiction. So β_j^i 's are distinct. Hence $\{a_{\beta_1^1}, a_{\beta_2^1}, \dots, a_{\beta_{n_1}^1}, a_{\beta_1^2}, a_{\beta_2^2}, \dots, a_{\beta_{n_2}^2}, \dots, a_{\beta_1^k}, a_{\beta_2^k}, \dots, a_{\beta_{n_k}^k}\} = \{a_1, a_2, \dots, a_n\}$. If $a_{\beta_1^i} a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \in I_i$ for every $1 \leq i \leq k$, then

$$a_1 a_2 \cdots a_n = a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} a_{\beta_1^2} a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} \in L,$$

thus we are done. Therefore we may assume that $a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} \notin I_1$. Since I_1 is ϕ - n_1 -absorbing primary and

$$a_{\beta_1^1} a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} a_{\beta_1^2} a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} a_{n+1} = a_1 \cdots a_{n+1} \in I_1 \setminus \phi(I_1),$$

then we have $a_{\beta_1^2} a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} a_{n+1} \in P_1$. On the other hand $a_{\beta_1^2} a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} a_{n+1} \in P_2 \cap \cdots \cap P_k$. Consequently $a_{\beta_1^2} a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k} a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} a_{n+1} \in \sqrt{L}$, which is a contradiction. Similarly $a_{\beta_1^i} a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \in I_i$ for every $2 \leq i \leq k$. Then $a_1 a_2 \cdots a_n \in L$.

(2) The proof is similar to that of part (1). \square

Corollary 2.9. *Let R be a ring with $1 \neq 0$ and let P_1, P_2, \dots, P_n be prime ideals of R . Suppose that for every $1 \leq i \leq n$, $P_i^{t_i}$ is a P_i -primary ideal of R where t_i is a positive integer. Then $P_1^{t_1} \cap P_2^{t_2} \cap \cdots \cap P_n^{t_n}$ and $P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n}$ are n -absorbing primary ideals of R . In particular, $P_1 \cap P_2 \cap \cdots \cap P_n$ and $P_1 P_2 \cdots P_n$ are n -absorbing primary ideals of R .*

Example 2.10. Let $R = \mathbb{Z}[X_2, X_3, \dots, X_n] + 3X_1\mathbb{Z}[X_2, X_3, \dots, X_n, X_1]$. Set $P_i := X_{i+1}R$ for $1 \leq i \leq n-1$ and $P_n := 3X_1\mathbb{Z}[X_2, X_3, \dots, X_n, X_1]$. Note that for every $1 \leq i \leq n$, P_i is a prime ideal of R . Let $I = P_1 P_2 \cdots P_{n-1} P_n^2$. Then $3X_1^2 \cdot X_2 \cdots X_n \cdot 3 = 9X_1^2 X_2 \cdots X_n \in I$ and $3X_1^2 \cdot X_2 \cdots X_n = 3X_1^2 X_2 \cdots X_n \notin I$. On the other hand $X_2 \cdots X_n \cdot 3 = 3X_2 \cdots X_n \notin \sqrt{I} \subseteq P_n$ and $3X_1^2 \cdot X_2 \cdots \widehat{X_i} \cdots X_n \cdot 3 = 9X_1^2 X_2 \cdots \widehat{X_i} \cdots X_n \notin \sqrt{I} \subseteq P_{i-1}$ for every $2 \leq i \leq n$. Hence I is not n -absorbing primary.

In [7, Example 2.7], the authors offered an example to show that if $I \subset J$ such that I is a 2-absorbing primary ideal of R and $\sqrt{I} = \sqrt{J}$, then J need not be a 2-absorbing ideal of R . They considered the ideal $J = \langle XYZ, Y^3, X^3 \rangle$ of the ring $R = \mathbb{Z}[X, Y, Z]$ and showed that $\sqrt{J} = \langle XY \rangle$. But $X \in \sqrt{J}$, which is a contradiction. Therefore their example is incorrect. In the following example we show that if $I \subset J$ such that I is a n -absorbing primary ideal of R and $\sqrt{I} = \sqrt{J}$, then J need not be a n -absorbing ideal of R .

Example 2.11. Let $R = K[X_1, X_2, \dots, X_{n+2}]$ where K is a field. Consider the ideal $J = \langle X_1 X_2 \cdots X_{n+1}, X_1^2 X_2 \cdots X_n, X_1^2 X_{n+2} \rangle$ of R . Then

$$\begin{aligned} \sqrt{J} &= \langle X_1 X_2 \cdots X_n, X_1 X_{n+2} \rangle \\ &= \langle X_1 \rangle \cap \langle X_2, X_{n+2} \rangle \cap \langle X_3, X_{n+2} \rangle \cap \cdots \cap \langle X_n, X_{n+2} \rangle. \end{aligned}$$

Set $P_1 = \langle X_1 \rangle$ and $P_i = \langle X_i, X_{n+2} \rangle$ for every $2 \leq i \leq n$. Note that P_i 's are prime ideals of R . Let $I = P_1^2 P_2 \cdots P_n$. Then $I \subset J$ and $\sqrt{I} = \sqrt{J} = \cap_{i=1}^n P_i$. By Corollary 2.9, I is an n -absorbing primary ideal of R , but J is not an n -absorbing primary ideal of R because $X_1 X_2 \cdots X_{n+1} \in J$, but $X_1 X_2 \cdots X_n \notin J$ and $X_2 \cdots X_{n+1} \notin \sqrt{J} \subseteq \langle X_1 \rangle$ and $X_1 \cdots \widehat{X_i} \cdots X_{n+1} \notin \sqrt{J} \subseteq \langle X_i, X_{n+2} \rangle$ for every $2 \leq i \leq n$.

Theorem 2.12. Let R be a ring, and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Suppose that I is an ideal of R such that $\sqrt{\phi(\sqrt{I})} \subseteq \phi(I)$. If \sqrt{I} is a ϕ -($n-1$)-absorbing ideal of R , then I is a ϕ - n -absorbing primary ideal of R .

Proof. Let \sqrt{I} be ϕ -($n-1$)-absorbing. Assume that $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some $a_1, a_2, \dots, a_{n+1} \in R$ and $a_1 a_2 \cdots a_n \notin I$. Since

$$(a_1 a_{n+1})(a_2 a_{n+1}) \cdots (a_n a_{n+1}) = (a_1 a_2 \cdots a_n) a_{n+1}^n \in I \subseteq \sqrt{I} \setminus \phi(\sqrt{I}).$$

Then for some $1 \leq i \leq n$,

$$(a_1 a_{n+1}) \cdots (\widehat{a_i a_{n+1}}) \cdots (a_n a_{n+1}) = (a_1 \cdots \widehat{a_i} \cdots a_n) a_{n+1}^{n-1} \in \sqrt{I},$$

and so $a_1 \cdots \widehat{a_i} \cdots a_n a_{n+1} \in \sqrt{I}$. Consequently I is ϕ - n -absorbing primary. \square

The following example gives an ideal J of a ring R where \sqrt{J} is an n -absorbing ideal of R , but J is not an n -absorbing primary ideal of R .

Example 2.13. Let $R = K[X_1, X_2, \dots, X_{n+2}]$ where K is a field and let $J = \langle X_1 X_2 \cdots X_{n+1}, X_1^2 X_2 \cdots X_n, X_1^2 X_{n+2} \rangle$. Then

$$\sqrt{J} = \langle X_1 \rangle \cap \langle X_2, X_{n+2} \rangle \cap \langle X_3, X_{n+2} \rangle \cap \cdots \cap \langle X_n, X_{n+2} \rangle.$$

By [2, Theorem 2.1(c)], \sqrt{J} is an n -absorbing ideal of R , but J is not an n -absorbing primary ideal of R as it is shown in Example 2.11.

We know that if I is an ideal of a ring R such that \sqrt{I} is a maximal ideal of R , then I is a primary ideal of R .

Theorem 2.14. Let I be an ideal of a ring R . If $\sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_n$ where M_i 's are maximal ideals of R , then I is an n -absorbing primary ideal of R .

Proof. Let $a_1 a_2 \cdots a_{n+1} \in I$ for some $a_1, a_2, \dots, a_{n+1} \in R$ such that $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin \sqrt{I}$ for every $1 \leq i \leq n$. If for some $1 \leq i \leq n$, $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in M_j$ (for every $1 \leq j \leq n$), then $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ and so we are done. Without loss of generality we may assume that for every $1 \leq i \leq n$, $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin M_i$, respectively. Since M_i 's are maximal, then $M_i + R(a_1 \cdots \widehat{a_i} \cdots a_{n+1}) = R$ for every $1 \leq i \leq n$. Therefore for every $1 \leq i \leq n$ there are $m_i \in M_i$ and $r_i \in R$ such that $m_i + r_i(a_1 \cdots \widehat{a_i} \cdots a_{n+1}) = 1$. So

$$\begin{aligned} m_1 m_2 \cdots m_n + \sum_{t=1}^n \sum_{\substack{\alpha_1=1 \\ \alpha_1 < \alpha_2 < \cdots < \alpha_t \leq n}}^{n-t+1} [r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_t} (m_1 \cdots \widehat{m_{\alpha_1}} \cdots \widehat{m_{\alpha_2}} \cdots \widehat{m_{\alpha_t}} \cdots m_n)] \\ \prod_{i=1}^t (a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1})] = 1 \end{aligned}$$

Since $m_1 m_2 \cdots m_n \in \sqrt{I}$, hence $(m_1 m_2 \cdots m_n)^t \in I$ for some $t \geq 1$. Thus

$$\begin{aligned} (m_1 m_2 \cdots m_n)^t + s \left[\sum_{t=1}^n \sum_{\substack{\alpha_1=1 \\ \alpha_1 < \alpha_2 < \cdots < \alpha_t \leq n}}^{n-t+1} [r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_t} (m_1 \cdots \widehat{m_{\alpha_1}} \cdots \widehat{m_{\alpha_2}} \cdots \widehat{m_{\alpha_t}} \cdots m_n)] \right. \\ \left. \prod_{i=1}^t (a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1}) \right] = 1 \end{aligned}$$

for some $s \in R$. Multiply $a_1 a_2 \cdots a_n$ on both sides to get

$$\begin{aligned} a_1 a_2 \cdots a_n &= a_1 a_2 \cdots a_n (m_1 m_2 \cdots m_n)^t + \\ &\quad s \left[\sum_{t=1}^n \sum_{\substack{\alpha_1=1 \\ \alpha_1 < \alpha_2 < \\ \cdots < \alpha_t \leq n}}^{n-t+1} [r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_t} (m_1 \cdots \widehat{m_{\alpha_1}} \cdots \widehat{m_{\alpha_2}} \cdots \widehat{m_{\alpha_t}} \cdots m_n) \right. \\ &\quad \left. (a_1 a_2 \cdots a_n) \prod_{i=1}^t (a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1}) \right] \in I. \end{aligned}$$

Hence I is an n -absorbing primary ideal. \square

Let R be an integral domain with $1 \neq 0$ and let K be the quotient field of R . A nonzero ideal I of R is said to be *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in K \mid xI \subseteq R\}$. An integral domain R is said to be a *Dedekind domain* if every nonzero proper ideal of R is invertible.

Theorem 2.15. *Let R be a Noetherian integral domain with $1 \neq 0$ that is not a field. The following conditions are equivalent:*

- (1) R is a Dedekind domain;
- (2) A nonzero proper ideal I of R is an n -absorbing primary ideal of R if and only if $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals M_1, M_2, \dots, M_i of R and some positive integers t_1, t_2, \dots, t_i ;
- (3) If I is a nonzero n -absorbing primary ideal of R , then $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals M_1, M_2, \dots, M_i of R and some positive integers t_1, t_2, \dots, t_i ;
- (4) A nonzero proper ideal I of R is an n -absorbing primary ideal of R if and only if $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct prime ideals P_1, P_2, \dots, P_i of R and some positive integers t_1, t_2, \dots, t_i ;
- (5) If I is a nonzero n -absorbing primary ideal of R , then $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct prime ideals P_1, P_2, \dots, P_i of R and some positive integers t_1, t_2, \dots, t_i .

Proof. (1) \Rightarrow (2) Assume that R is a Dedekind domain that is not a field. Then every nonzero prime ideal of R is maximal. Let I be a nonzero n -absorbing primary ideal of R . Since R is a Dedekind domain, then there are distinct maximal ideals M_1, M_2, \dots, M_i of R ($k \geq 1$) such that $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ in which t_j 's are positive integers. Therefore $\sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_i$. Since I is n -absorbing primary and every prime ideal of R is maximal, then \sqrt{I} is the intersection of at most n maximal ideals of R , by Corollary 2.7. So $i \leq n$.

Conversely, suppose that $I = M_1^{t_1} M_2^{t_2} \cdots M_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct maximal ideals M_1, M_2, \dots, M_i of R and some positive integers t_1, t_2, \dots, t_i . Then I is n -absorbing primary, by Corollary 2.9.

(1) \Rightarrow (4) The proof is similar to that of (1) \Rightarrow (2).

(2) \Rightarrow (3), (3) \Rightarrow (5) and (4) \Rightarrow (5) are evident.

(5) \Rightarrow (1) Let M be an arbitrary maximal ideal of R and I be an ideal of R such that $M^2 \subsetneq I \subsetneq M$. Hence $\sqrt{I} = M$ and so I is M -primary. Then I is n -absorbing primary, and thus by part (5) we have that $I = P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$ for some $1 \leq i \leq n$ and some distinct prime ideals P_1, P_2, \dots, P_i of R and some positive integers t_1, t_2, \dots, t_i . Then $\sqrt{I} = P_1 \cap P_2 \cap \cdots \cap P_i = M$ which shows that I is a

power of M , a contradiction. Therefore, there are no ideals properly between M^2 and M . Consequently R is a Dedekind domain, by [13, Theorem 39.2, p. 470]. \square

Since every principal ideal domain is a Dedekind domain, we have the following result as a consequence of Theorem 2.15.

Corollary 2.16. *Let R be a principal ideal domain and I be a nonzero proper ideal of R . Then I is an n -absorbing primary ideal of R if and only if $I = R(p_1^{t_1} p_2^{t_2} \cdots p_i^{t_i})$, where p_j 's are prime elements of R , $1 \leq i \leq n$ and t_j 's are some integers.*

The following example shows that an n -absorbing primary ideal of a ring R need not be of the form $P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$, where P_j 's are prime ideals of R , $1 \leq i \leq n$ and t_j 's are some integers.

Example 2.17. Let $R = K[X_1, X_2, \dots, X_n]$ where K is a field and let $I = \langle X_1, X_2, \dots, X_{n-1}, X_n^2 \rangle$. Since I is $\langle X_1, X_2, \dots, X_n \rangle$ -primary, then I is an n -absorbing primary ideal of R . But I is not in the form of $P_1^{t_1} P_2^{t_2} \cdots P_i^{t_i}$, where P_j 's are prime ideals of R , $1 \leq i \leq n$ and t_j 's are some integers.

Theorem 2.18. *Let R be a ring, $a \in R$ a nonunit and $m \geq 2$ a positive integer. If $(0 :_R a) \subseteq \langle a \rangle$, then $\langle a \rangle$ is ϕ - n -absorbing primary, for some ϕ with $\phi \leq \phi_m$ if and only if $\langle a \rangle$ is n -absorbing primary.*

Proof. We may assume that $\langle a \rangle$ is ϕ_m - n -absorbing primary. Let $x_1 x_2 \cdots x_{n+1} \in \langle a \rangle$ for some $x_1, x_2, \dots, x_{n+1} \in R$. If $x_1 x_2 \cdots x_{n+1} \notin \langle a^m \rangle$, then either $x_1 x_2 \cdots x_n \in \langle a \rangle$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{\langle a \rangle}$ for some $1 \leq i \leq n$. Therefore, assume that $x_1 x_2 \cdots x_{n+1} \in \langle a^m \rangle$. Hence $x_1 x_2 \cdots x_n (x_{n+1} + a) \in \langle a \rangle$. If $x_1 x_2 \cdots x_n (x_{n+1} + a) \notin \langle a^m \rangle$, then either $x_1 x_2 \cdots x_n \in \langle a \rangle$ or $x_1 \cdots \widehat{x_i} \cdots x_n (x_{n+1} + a) \in \sqrt{\langle a \rangle}$ for some $1 \leq i \leq n$. So, either $x_1 x_2 \cdots x_n \in \langle a \rangle$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{\langle a \rangle}$ for some $1 \leq i \leq n$. Hence, suppose that $x_1 x_2 \cdots x_n (x_{n+1} + a) \in \langle a^m \rangle$. Thus $x_1 x_2 \cdots x_{n+1} \in \langle a^m \rangle$ implies that $x_1 x_2 \cdots x_n a \in \langle a^m \rangle$. Therefore, there exists $r \in R$ such that $x_1 x_2 \cdots x_n - r a^{m-1} \in (0 :_R a) \subseteq \langle a \rangle$. Consequently $x_1 x_2 \cdots x_n \in \langle a \rangle$. \square

Corollary 2.19. *Let R be an integral domain, $a \in R$ a nonunit element and $m \geq 2$ a positive integer. Then $\langle a \rangle$ is ϕ - n -absorbing primary, for some ϕ with $\phi \leq \phi_m$ if and only if $\langle a \rangle$ is n -absorbing primary.*

Theorem 2.20. *Let V be a valuation domain and n be a natural number. Suppose that I is an ideal of V such that I^{n+1} is not principal. Then I is a ϕ_{n+1} - n -absorbing primary if and only if it is n -absorbing primary.*

Proof. (\Rightarrow) Assume that I is ϕ_n - n -absorbing primary that is not n -absorbing primary. Therefore there are $a_1, \dots, a_{n+1} \in R$ such that $a_1 \cdots a_{n+1} \in I$, but neither $a_1 \cdots a_n \in I$ nor $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for every $1 \leq i \leq n$. Hence $\langle a_i \rangle \not\subseteq I$ for every $1 \leq i \leq n+1$. Since V is a valuation domain, thus $I \subset \langle a_i \rangle$ for every $1 \leq i \leq n+1$, and so $I^{n+1} \subseteq \langle a_1 \cdots a_{n+1} \rangle$. Since I^{n+1} is not principal, then $a_1 \cdots a_{n+1} \in I \setminus I^{n+1}$. Therefore I ϕ_{n+1} - n -absorbing primary implies that either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, which is a contradiction. Consequently I is n -absorbing primary.

(\Leftarrow) is trivial. \square

Let J be an ideal of R and $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Define $\phi_J : \mathfrak{J}(R/J) \rightarrow \mathfrak{J}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal $I \in \mathfrak{J}(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$).

Theorem 2.21. *Let $J \subseteq I$ be proper ideals of a ring R , and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function.*

- (1) *If I is a ϕ - n -absorbing primary ideal of R , then I/J is a ϕ_J - n -absorbing primary ideal of R/J .*
- (2) *If $J \subseteq \phi(I)$ and I/J is a ϕ_J - n -absorbing primary ideal of R/J , then I is a ϕ - n -absorbing primary ideal of R .*
- (3) *If $\phi(I) \subseteq J$ and I is a ϕ - n -absorbing primary ideal of R , then I/J is a weakly n -absorbing primary ideal of R/J .*
- (4) *If $\phi(J) \subseteq \phi(I)$, J is a ϕ - n -absorbing primary ideal of R and I/J is a weakly n -absorbing primary ideal of R/J , then I is a ϕ - n -absorbing primary ideal of R .*

Proof. (1) Let $a_1, a_2, \dots, a_{n+1} \in R$ be such that $(a_1 + J)(a_2 + J) \cdots (a_{n+1} + J) \in (I/J) \setminus \phi_J(I/J) = (I/J) \setminus (\phi(I) + J)/J$. Then $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ and I ϕ - n -absorbing primary gives either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Therefore either $(a_1 + J) \cdots (a_n + J) \in I/J$ or $(a_1 + J) \cdots (\widehat{a_i + J}) \cdots (a_{n+1} + J) \in \sqrt{I}/J = \sqrt{I/J}$ for some $1 \leq i \leq n$. This shows that I/J is ϕ_J - n -absorbing primary.

(2) Suppose that $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some $a_1, a_2, \dots, a_{n+1} \in R$. Then $(a_1 + J)(a_2 + J) \cdots (a_{n+1} + J) \in (I/J) \setminus (\phi(I) + J)/J = (I/J) \setminus \phi_J(I/J)$. Since I/J is assumed to be ϕ_J - n -absorbing primary, we get either $(a_1 + J) \cdots (a_n + J) \in I/J$ or $(a_1 + J) \cdots (\widehat{a_i + J}) \cdots (a_{n+1} + J) \in \sqrt{I/J} = \sqrt{I}/J$ for some $1 \leq i \leq n$. Consequently, either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, that I is ϕ - n -absorbing primary.

(3) is a direct consequence of part (1).

(4) Let $a_1 \cdots a_{n+1} \in I \setminus \phi(I)$ where $a_1, \dots, a_{n+1} \in R$. Note that $a_1 \cdots a_{n+1} \notin \phi(J)$ because $\phi(J) \subseteq \phi(I)$. If $a_1 \cdots a_{n+1} \in J$, then either $a_1 \cdots a_n \in J \subseteq I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{J} \subseteq \sqrt{I}$ for some $1 \leq i \leq n$, since J is ϕ - n -absorbing primary. If $a_1 \cdots a_{n+1} \notin J$, then $(a_1 + I) \cdots (a_{n+1} + I) \in (I/J) \setminus \{0\}$ and so either $(a_1 + I) \cdots (a_n + I) \in I/J$ or $(a_1 + I) \cdots (\widehat{a_i + I}) \cdots (a_{n+1} + I) \in \sqrt{I/J} = \sqrt{I}/J$ for some $1 \leq i \leq n$. Therefore, either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Consequently I is a ϕ - n -absorbing primary ideal of R . \square

Corollary 2.22. *Let R be a ring, and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. An ideal I of R is ϕ - n -absorbing primary if and only if $I/\phi(I)$ is a weakly n -absorbing primary ideal of $R/\phi(I)$.*

Proof. In parts (2) and (3) of Theorem 2.21 set $J = \phi(I)$. \square

Corollary 2.23. *Let R be a ring, $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ a function and I a proper ideal of R such that $\phi(\langle I, X \rangle) \subseteq \langle X \rangle$. Then $\langle I, X \rangle$ is a ϕ - n -absorbing primary ideal of $R[X]$ if and only if I is a weakly n -absorbing ideal of R .*

Proof. By parts (3), (4) of Theorem 2.21 and regarding the isomorphism $\langle I, X \rangle / \langle X \rangle \simeq I$ in $R[X] / \langle X \rangle \simeq R$ we have the result. \square

Let S be a multiplicatively closed subset of a ring R . Let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function and define $\phi_S : \mathfrak{J}(R_S) \rightarrow \mathfrak{J}(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = (\phi(J \cap R))_S$ (and $\phi_S(J) = \emptyset$ if $\phi(J \cap R) = \emptyset$) for every ideal J of R_S . Note that $\phi_S(J) \subseteq J$.

Let M be an R -module. The set of all zero divisors on M is:

$$Z_R(M) = \{r \in R \mid \text{there exists an element } 0 \neq x \in M \text{ such that } rx = 0\}.$$

Proposition 2.24. *Let R be a ring and $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Suppose that S is a multiplicatively closed subset of R and I is a proper ideal of R .*

- (1) *If I is a ϕ - n -absorbing primary ideal of R with $I \cap S = \emptyset$ and $\phi(I)_S \subseteq \phi_S(I_S)$, then I_S is a ϕ_S - n -absorbing primary ideal of R_S .*
- (2) *If I_S is a ϕ_S - n -absorbing primary ideal of R_S with $\phi_S(I_S) \subseteq \phi(I)_S$, $S \cap Z_R(\frac{I}{\phi(I)}) = \emptyset$ and $S \cap Z_R(\frac{R}{I}) = \emptyset$, then I is a ϕ - n -absorbing primary ideal of R .*

Proof. (1) Assume that $\frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_{n+1}}{s_{n+1}} \in I_S \setminus \phi_S(I_S)$ for some $\frac{a_1}{s_1}, \frac{a_2}{s_2}, \dots, \frac{a_{n+1}}{s_{n+1}} \in R_S$ such that $\frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \notin I_S$. Since $\frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_{n+1}}{s_{n+1}} \in I_S$, then there is $s \in S$ such that $sa_1a_2 \cdots a_{n+1} \in I$. If $sa_1a_2 \cdots a_{n+1} \in \phi(I)$, then $\frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_{n+1}}{s_{n+1}} = \frac{sa_1a_2 \cdots a_{n+1}}{ss_1s_2 \cdots s_{n+1}} \in \phi(I)_S \subseteq \phi_S(I_S)$, a contradiction. Hence $a_1a_2 \cdots a_n(sa_{n+1}) \in I \setminus \phi(I)$. As I is ϕ - n -absorbing primary, we get either $a_1a_2 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_n(sa_{n+1}) \in \sqrt{I}$ for some $1 \leq i \leq n$. The first case implies that $\frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in I_S$ which is a contradiction, and the second case implies that $\frac{a_1}{s_1} \cdots \frac{\widehat{a_i}}{s_i} \cdots \frac{a_{n+1}}{s_{n+1}} \in (\sqrt{I})_S = \sqrt{I}_S$ for some $1 \leq i \leq n$. Consequently I_S is a ϕ_S - n -absorbing primary ideal of R_S .

(2) Let $a_1a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some $a_1, a_2, \dots, a_{n+1} \in R$ and let $a_1a_2 \cdots a_n \notin I$. Then $\frac{a_1}{1} \frac{a_2}{1} \cdots \frac{a_{n+1}}{1} \in I_S$. Assume that $\frac{a_1}{1} \frac{a_2}{1} \cdots \frac{a_{n+1}}{1} \in \phi_S(I_S)$. Since $\phi_S(I_S) \subseteq \phi(I)_S$, then there exists a $s \in S$ such that $sa_1a_2 \cdots a_{n+1} \in \phi(I)$. Since $S \cap Z_R(\frac{I}{\phi(I)}) = \emptyset$ we have that $a_1a_2 \cdots a_{n+1} \in \phi(I)$, which is a contradiction. Therefore $\frac{a_1}{1} \frac{a_2}{1} \cdots \frac{a_{n+1}}{1} \in I_S \setminus \phi_S(I_S)$. Hence, either $\frac{a_1}{1} \frac{a_2}{1} \cdots \frac{a_n}{1} \in I_S$ or $\frac{a_1}{1} \cdots \frac{\widehat{a_i}}{1} \cdots \frac{a_{n+1}}{1} \in \sqrt{I}_S = (\sqrt{I})_S$ for some $1 \leq i \leq n$. If $\frac{a_1}{1} \frac{a_2}{1} \cdots \frac{a_n}{1} \in I_S$, then there exists $u \in S$ such that $ua_1a_2 \cdots a_n \in I$ and so the assumption $S \cap Z_R(\frac{R}{I}) = \emptyset$ shows that $a_1a_2 \cdots a_n \in I$, a contradiction. Therefore, there is $1 \leq i \leq n$ such that $\frac{a_1}{1} \cdots \frac{\widehat{a_i}}{1} \cdots \frac{a_{n+1}}{1} \in (\sqrt{I})_S$, and thus there is a $t \in S$ such that $ta_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$. Note that $S \cap Z_R(\frac{R}{I}) = \emptyset$ implies that $S \cap Z_R(\frac{R}{\sqrt{I}}) = \emptyset$, then $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$. Consequently I is a ϕ - n -absorbing primary ideal of R . \square

Let $f : R \rightarrow T$ be a homomorphism of rings and let $\phi_T : \mathfrak{J}(T) \rightarrow \mathfrak{J}(T) \cup \{\emptyset\}$ be a function. Define $\phi_R : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ by $\phi_R(I) = \phi_T(I^e)^c$ (and $\phi_R(I) = \emptyset$ if $\phi_T(I^e) = \emptyset$). We recall that if R is a Prüfer domain or $T = R_S$ for some multiplicatively closed subset S of R , then for every ideal J of T we have $J^{ce} = J$.

Theorem 2.25. *Let $f : R \rightarrow T$ be a homomorphism of rings. If J is a ϕ_T - n -absorbing primary ideal of T such that $\phi_T(J) \subseteq \phi_T(J^{ce})$ (e.g. where $J = J^{ce}$), then J^c is a ϕ_R - n -absorbing primary ideal of R .*

Proof. Let $a_1a_2 \cdots a_{n+1} \in J^c \setminus \phi_R(J^c)$ for some $a_1, a_2, \dots, a_{n+1} \in R$. If $f(a_1)f(a_2) \cdots f(a_{n+1}) \in \phi_T(J)$, then $a_1a_2 \cdots a_{n+1} \in \phi_T(J)^c \subseteq \phi_T(J^{ce})^c = \phi_R(J^c)$, which is a contradiction. Therefore $f(a_1)f(a_2) \cdots f(a_{n+1}) \in J \setminus \phi_T(J)$. Hence, either $f(a_1)f(a_2) \cdots f(a_n) \in J$ or $f(a_1) \cdots \widehat{f(a_i)} \cdots f(a_{n+1}) \in \sqrt{J}$ for some $1 \leq i \leq n$. Thus, either $a_1a_2 \cdots a_n \in J^c$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{J^c}$ for some $1 \leq i \leq n$. Consequently J^c is a ϕ_R - n -absorbing primary ideal of R . \square

Let R, T be rings and $\psi_R : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Define $\psi_T : \mathfrak{J}(T) \rightarrow \mathfrak{J}(T) \cup \{\emptyset\}$ by $\psi_T(J) = \psi_R(J^c)^e$ (and $\psi_T(J) = \emptyset$ if $\psi_R(J^c) = \emptyset$). We recall that if $f : R \rightarrow T$ is a faithfully flat homomorphism of rings, then for every ideal I of R we have $I^{ec} = I$.

Theorem 2.26. *Let $f : R \rightarrow T$ be a faithfully flat homomorphism of rings.*

- (1) *If J is a ψ_T - n -absorbing primary ideal of T , then J^c is a ψ_R - n -absorbing primary ideal of R .*
- (2) *If I^e is a ψ_T - n -absorbing primary ideal of T for some ideal I of R , then I is a ψ_R - n -absorbing primary ideal of R .*

Proof. (1) Suppose that J is a ψ_T - n -absorbing primary ideal of T . In Theorem 2.25 get $\phi_T := \psi_T$. Let I be an ideal of R . Then

$$\phi_R(I) = \phi_T(I^e)^c = \psi_T(I^e)^c = \psi_R(I^{ec})^{ec} = \psi_R(I).$$

So $\phi_R = \psi_R$. Moreover, $\psi_T(J) = \psi_R(J^c)^e = \psi_R(J^{cec})^e = \psi_T(J^{ce})$. Therefore J^c is a ψ_R - n -absorbing primary ideal of R .

(2) By part (1). \square

Proposition 2.27. *Let I be an ideal of a ring R such that $\phi(I)$ be an n -absorbing primary ideal of R . If I is a ϕ - n -absorbing primary ideal of R , then I is an n -absorbing primary ideal of R .*

Proof. Assume that $a_1 a_2 \cdots a_{n+1} \in I$ for some elements $a_1, a_2, \dots, a_{n+1} \in R$ such that $a_1 a_2 \cdots a_n \notin I$. If $a_1 a_2 \cdots a_{n+1} \in \phi(I)$, then $\phi(I)$ n -absorbing primary and $a_1 a_2 \cdots a_n \notin \phi(I)$ implies that $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{\phi(I)} \subseteq \sqrt{I}$ for some $1 \leq i \leq n$, and so we are done. When $a_1 a_2 \cdots a_{n+1} \notin \phi(I)$ clearly the result follows. \square

We say that a ϕ -prime ideal P of a ring R is a divided ϕ -prime ideal if $P \subset xR$ for every $x \in R \setminus P$; thus a divided ϕ -prime ideal is comparable to every ideal of R .

Theorem 2.28. *Let P be a divided ϕ -prime ideal of a ring R . Suppose that I is a ϕ - n -absorbing ideal of R with $\sqrt{I} = P$ and $\phi(P) \subseteq \phi(I)$. Then I is a ϕ -primary ideal of R .*

Proof. Let $xy \in I \setminus \phi(I)$ for $x, y \in R$ and $y \notin P$. Since $xy \in P \setminus \phi(P)$, then $x \in P$. If $y^{n-1} \in \phi(P)$, then $y \in \sqrt{I} = P$, which is a contradiction. Therefore $y^{n-1} \notin \phi(P)$, and so $y^{n-1} \notin P$. Thus $P \subset y^{n-1}R$, because P is a divided ϕ -prime ideal of R . Hence $x = y^{n-1}z$ for some $z \in R$. As $y^n z = yx \in I \setminus \phi(I)$, $y^n \notin I$, and I is a ϕ - n -absorbing ideal of R , we have $x = y^{n-1}z \in I$. Hence I is a ϕ -primary ideal of R . \square

Let I be an ideal of a ring R and $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Assume that I is a ϕ - n -absorbing primary ideal of R and $a_1, \dots, a_{n+1} \in R$. We say that (a_1, \dots, a_{n+1}) is an ϕ -($n+1$)-tuple of I if $a_1 \cdots a_{n+1} \in \phi(I)$, $a_1 a_2 \cdots a_n \notin I$ and for each $1 \leq i \leq n$, $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin \sqrt{I}$.

In the following theorem $a_1 \cdots \widehat{a_i} \cdots \widehat{a_j} \cdots a_n$ denotes that a_i and a_j are eliminated from $a_1 \cdots a_n$.

Theorem 2.29. *Let I be a ϕ - n -absorbing primary ideal of a ring R and suppose that (a_1, \dots, a_{n+1}) is a ϕ -($n+1$)-tuple of I for some $a_1, \dots, a_{n+1} \in R$. Then for every elements $\alpha_1, \alpha_2, \dots, \alpha_m \in \{1, 2, \dots, n+1\}$ which $1 \leq m \leq n$,*

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1} I^m \subseteq \phi(I).$$

Proof. We use induction on m . Let $m = 1$ and suppose that $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1}x \notin \phi(I)$ for some $x \in I$. Then $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1}(a_{\alpha_1} + x) \notin \phi(I)$. Since I is a ϕ - n -absorbing primary ideal of R and $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} \notin I$, we conclude that $a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots a_{n+1}(a_{\alpha_1} + x) \in \sqrt{I}$, for some $1 \leq \alpha_2 \leq n+1$ different from α_1 . Hence $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1} \in \sqrt{I}$, a contradiction. Thus $a_1 \cdots \widehat{a_{\alpha_1}} \cdots a_{n+1}I \subseteq \phi(I)$. Now suppose $m > 1$ and assume that for all integers less than m the claim holds. Let $a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}x_1x_2 \cdots x_m \notin \phi(I)$ for some $x_1, x_2, \dots, x_m \in I$. By induction hypothesis, we conclude that there exists $\zeta \in \phi(I)$ such that

$$\begin{aligned} a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \\ = \zeta + a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}x_1x_2 \cdots x_m \notin \phi(I). \end{aligned}$$

Now, we consider two cases.

Case 1. Assume that $\alpha_m < n+1$. Since I is ϕ - n -absorbing primary, then either

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_n(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in I,$$

or

$$\begin{aligned} a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots \widehat{a_j} \cdots a_{n+1}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \\ \in \sqrt{I}, \end{aligned}$$

for some $j < n+1$ distinct from α_i 's; or

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}(a_{\alpha_1} + x_1) \cdots (\widehat{a_{\alpha_i} + x_i}) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$

for some $1 \leq i \leq m$. Thus either $a_1a_2 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_j} \cdots a_{n+1} \in \sqrt{I}$ or $a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1} \in \sqrt{I}$, which any of these cases has a contradiction.

Case 2. Assume that $\alpha_m = n+1$. Since I is ϕ - n -absorbing primary, then either

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_{n+1}}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (\widehat{a_{\alpha_m} + x_m}) \in I,$$

or

$$\begin{aligned} a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_j} \cdots \widehat{a_{n+1}}(a_{\alpha_1} + x_1)(a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \\ \in \sqrt{I}, \end{aligned}$$

for some $j < n+1$ different from α_i 's; or

$$\begin{aligned} a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_{m-1}}} \cdots \widehat{a_{n+1}}(a_{\alpha_1} + x_1) \cdots (\widehat{a_{\alpha_i} + x_i}) \cdots (a_{\alpha_m} + x_m) \\ \in \sqrt{I} \end{aligned}$$

for some $1 \leq i \leq m-1$. Thus either $a_1a_2 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_j} \cdots a_{n+1} \in \sqrt{I}$ or $a_1 \cdots \widehat{a_{\alpha_i}} \cdots a_{n+1} \in \sqrt{I}$, which any of these cases has a contradiction. Thus

$$a_1 \cdots \widehat{a_{\alpha_1}} \cdots \widehat{a_{\alpha_2}} \cdots \widehat{a_{\alpha_m}} \cdots a_{n+1}I^m \subseteq \phi(I).$$

□

Theorem 2.30. *Let I be an ϕ - n -absorbing primary ideal of R that is not an n -absorbing primary ideal. Then*

- (1) $I^{n+1} \subseteq \phi(I)$.
- (2) $\sqrt{I} = \sqrt{\phi(I)}$.

Proof. (1) Since I is not an n -absorbing primary ideal of R , I has an ϕ -($n+1$)-triple-zero (a_1, \dots, a_{n+1}) for some $a_1, \dots, a_{n+1} \in R$. Suppose that $x_1 x_2 \cdots x_{n+1} \notin \phi(I)$ for some $x_1, x_2, \dots, x_{n+1} \in I$. Then by Theorem 2.29, there is $\zeta \in \phi(I)$ such that $(a_1 + x_1) \cdots (a_{n+1} + x_{n+1}) = \zeta + x_1 x_2 \cdots x_{n+1} \notin \phi(I)$. Hence either $(a_1 + x_1) \cdots (a_n + x_n) \in I$ or $(a_1 + x_1) \cdots (\widehat{a_i + x_i}) \cdots (a_{n+1} + x_{n+1}) \in \sqrt{I}$ for some $1 \leq i \leq n$. Thus either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, a contradiction. Hence $I^{n+1} \subseteq \phi(I)$.

(2) Clearly, $\sqrt{\phi(I)} \subseteq \sqrt{I}$. As $I^{n+1} \subseteq \phi(I)$, we get $\sqrt{I} \subseteq \sqrt{\phi(I)}$, as required. \square

Corollary 2.31. *Let I be an ideal of a ring R that is not n -absorbing primary.*

- (1) *If I is weakly n -absorbing primary, then $I^{n+1} = \{0\}$ and $\sqrt{I} = \text{Nil}(R)$.*
- (2) *If I is ϕ - n -absorbing primary where $\phi \leq \phi_{n+2}$, then $I^{n+1} = I^{n+2}$.*

Corollary 2.32. *Let I be a ϕ - n -absorbing primary ideal where $\phi \leq \phi_{n+2}$. Then I is ω - n -absorbing primary.*

Proof. If I is n -absorbing primary, then it is ω - n -absorbing primary. So assume that I is not n -absorbing primary. Then $I^{n+1} = I^{n+2}$ by Corollary 2.31(2). By hypothesis I is ϕ - n -absorbing primary and $\phi \leq \phi_{n+1}$. So I is ϕ_{n+1} - n -absorbing primary. On the other hand $\phi_\omega(I) = I^{n+1} = \phi_{n+1}(I)$. Therefore I is ω - n -absorbing primary. \square

Theorem 2.33. *Let R be a ring and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Suppose that $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals of R such that for every $\lambda, \lambda' \in \Lambda$, $\sqrt{\phi(I_\lambda)} = \sqrt{\phi(I_{\lambda'})}$ and $\phi(I_\lambda) \subseteq \phi(I)$. If for every $\lambda \in \Lambda$, I_λ is a ϕ - n -absorbing primary ideal of R that is not n -absorbing primary, then $I = \bigcap_{\lambda \in \Lambda} I_\lambda$ is a ϕ - n -absorbing primary ideal of R .*

Proof. Since I_λ 's are ϕ - n -absorbing primary but are not n -absorbing primary, then for every $\lambda \in \Lambda$, $\sqrt{I_\lambda} = \sqrt{\phi(I_\lambda)}$, by Theorem 2.30. On the other hand $\phi(I_\lambda) \subseteq \phi(I)$ for every $\lambda \in \Lambda$, and so $\sqrt{\phi(I_\lambda)} \subseteq \sqrt{I}$. Hence $\sqrt{I} = \sqrt{I_\lambda} = \sqrt{\phi(I_\lambda)}$ for every $\lambda \in \Lambda$. Let $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some $a_1, a_2, \dots, a_{n+1} \in R$, and let $a_1 a_2 \cdots a_n \notin I$. Therefore there is a $\lambda \in \Lambda$ such that $a_1 a_2 \cdots a_n \notin I_\lambda$. Since I_λ is ϕ - n -absorbing primary and $a_1 a_2 \cdots a_{n+1} \in I_\lambda \setminus \phi(I_\lambda)$, then $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I_\lambda} = \sqrt{I}$ for some $1 \leq i \leq n$. Consequently I is a ϕ - n -absorbing primary ideal of R . \square

Corollary 2.34. *Let R be a ring, $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function and I be an ideal of R . Suppose that $\sqrt{\phi(I)} = \phi(\sqrt{I})$ that is an n -absorbing ideal of R . If I is a ϕ - n -absorbing primary ideal of R , then \sqrt{I} is an n -absorbing ideal of R .*

Proof. Assume that I is a ϕ - n -absorbing primary ideal of R . If I is an n -absorbing primary ideal of R , then \sqrt{I} is an n -absorbing ideal, by Theorem 2.6. If I is not an n -absorbing primary ideal of R , then by Theorem 2.30 and by our hypothesis, $\sqrt{I} = \sqrt{\phi(I)}$ which is an n -absorbing ideal. \square

Theorem 2.35. *Let I be a ϕ - n -absorbing primary ideal of a ring R that is not n -absorbing primary and let J be a ϕ - m -absorbing primary ideal of R that is not m -absorbing primary, and $n \geq m$. Suppose that the two ideals $\phi(I)$ and $\phi(J)$ are not coprime. Then*

- (1) $\sqrt{I+J} = \sqrt{\phi(I) + \phi(J)}$.
- (2) *If $\phi(I) \subseteq J$ and $\phi(J) \subseteq \phi(I+J)$, then $I+J$ is a ϕ - n -absorbing primary ideal of R .*

Proof. (1) By Theorem 2.30, we have $\sqrt{I} = \sqrt{\phi(I)}$ and $\sqrt{J} = \sqrt{\phi(J)}$. Now, by [21, 2.25(i)] the result follows.

(2) Assume that $\phi(I) \subseteq J$ and $\phi(J) \subseteq \phi(I + J)$. Since $\phi(I) + \phi(J) \neq R$, then $I + J$ is a proper ideal of R , by part (1). Since $(I + J)/J \simeq I/(I \cap J)$ and I is ϕ - n -absorbing primary, we get that $(I + J)/J$ is a weakly n -absorbing primary ideal of R/J , by Theorem 2.21(3). On the other hand J is also ϕ - n -absorbing primary, by Remark 2.1(6). Now, the assertion follows from Theorem 2.21(4). \square

Let R be a ring and M an R -module. A submodule N of M is called a pure submodule if the sequence $0 \rightarrow N \otimes_R E \rightarrow M \otimes_R E$ is exact for every R -module E .

As another consequence of Theorem 2.30 we have the following corollary.

Corollary 2.36. *Let R be a ring.*

- (1) *If I is a pure ϕ - n -absorbing primary ideal of R that is not n -absorbing primary, then $I = \phi(I)$.*
- (2) *If R is von Neumann regular ring, then every ϕ - n -absorbing primary ideal of R that is not n -absorbing primary is of the form $\phi(I)$ for some ideal I of R .*

Proof. Note that every pure ideal is idempotent (see [12]), also every ideal of a von Neumann regular ring is idempotent. \square

Theorem 2.37. *Let R be a ring and $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Let I be a ϕ -($n-1$)-absorbing primary ideal of R that is not $(n-1)$ -absorbing primary, and J be an ideal of R such that $J \subseteq I$ with $\phi(I) \subseteq \phi(J)$. Then J is a ϕ - n -absorbing primary ideal of R .*

Proof. Since I is a ϕ -($n-1$)-absorbing primary ideal that is not $(n-1)$ -absorbing primary we have $\sqrt{I} = \sqrt{\phi(I)}$, by Theorem 2.30. Hence $\sqrt{J} = \sqrt{I} = \sqrt{\phi(I)}$. Let $a_1 a_2 \cdots a_{n+1} \in J \setminus \phi(J)$ for some $a_1, a_2, \dots, a_{n+1} \in R$ such that $a_1 a_2 \cdots a_n \notin J$. Since $J \subseteq I$, we have $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$. Consider two cases.

Case 1. Assume that $a_1 a_2 \cdots a_n \notin I$. Since I is ϕ -($n-1$)-absorbing primary, then it is ϕ - n -absorbing primary, by Remark 2.1(6). Hence $a_1 \cdots \hat{a}_i \cdots a_{n+1} \in \sqrt{I} = \sqrt{J}$ for some $1 \leq i \leq n$.

Case 2. Assume that $a_1 a_2 \cdots a_n \in I$. Since $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$, we have that $a_1 a_2 \cdots a_n \in I \setminus \phi(I)$. On the other hand I is a ϕ -($n-1$)-absorbing primary ideal, so either $a_1 a_2 \cdots a_{n-1} \in I \subseteq \sqrt{J}$ or $a_1 \cdots \hat{a}_i \cdots a_n \in \sqrt{I} = \sqrt{J}$ for some $1 \leq i \leq n-1$. Hence $a_1 \cdots \hat{a}_i \cdots a_{n+1} \in \sqrt{J}$ for some $1 \leq i \leq n$. Consequently J is a ϕ - n -absorbing primary ideal of R . \square

3. ϕ - n -ABSORBING PRIMARY IDEALS IN DIRECT PRODUCTS OF COMMUTATIVE RINGS

Theorem 3.1. *Let R_1 and R_2 be rings, and let I be a weakly n -absorbing primary ideal of R_1 . Then $J = I \times R_2$ is a ϕ - n -absorbing primary ideal of $R = R_1 \times R_2$ for each ϕ with $\phi_\omega \leq \phi \leq \phi_1$.*

Proof. Suppose that I is a weakly n -absorbing primary ideal of R_1 . If I is n -absorbing primary, then J is n -absorbing primary and hence is ϕ - n -absorbing primary, for all ϕ . Assume that I is not n -absorbing primary. Then $I^{n+1} = \{0\}$, Corollary 2.31(1). Hence $J^{n+1} = \{0\} \times R_2$ and hence $\phi_\omega(J) = \{0\} \times R_2$. Therefore, $J \setminus \phi_\omega(J) = (I \setminus \{0\}) \times R_2$. Let $(x_1, y_1)(x_2, y_2) \cdots (x_{n+1}, y_{n+1}) \in$

$J \setminus \phi_\omega(J)$ for some $x_1, x_2, \dots, x_{n+1} \in R_1$ and $y_1, y_2, \dots, y_{n+1} \in R_2$. Then clearly $x_1 x_2 \cdots x_{n+1} \in I \setminus \{0\}$. Since I is weakly n -absorbing primary, either $x_1 \cdots x_n \in I$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Therefore, either $(x_1, y_1) \cdots (x_n, y_n) \in J = I \times R_2$ or $(x_1, y_1) \cdots \widehat{(x_i, y_i)} \cdots (x_{n+1}, y_{n+1}) \in \sqrt{J} = \sqrt{I} \times R_2$ for some $1 \leq i \leq n$. Consequently J is a ω - n -absorbing primary and hence ϕ - n -absorbing primary. \square

Theorem 3.2. *Let R be a ring and J be a finitely generated proper ideal of R . Suppose that J is ϕ - n -absorbing primary, where $\phi \leq \phi_{n+2}$. Then, either J is weakly n -absorbing primary or $J^{n+1} \neq 0$ is idempotent and R decomposes as $R_1 \times R_2$ where $R_2 = J^{n+1}$ and $J = I \times R_2$, where I is weakly n -absorbing primary.*

Proof. If J is n -absorbing primary, then J is weakly n -absorbing primary. So we can assume that J is not n -absorbing primary. Then by Corollary 2.31(2), $J^{n+1} = J^{n+2}$ and hence $J^{n+1} = J^{2(n+1)}$. Thus J^{n+1} is idempotent, since J^{n+1} is finitely generated, $J^{n+1} = \langle e \rangle$ for some idempotent element $e \in R$. Suppose $J^{n+1} = 0$. So $\phi(J) = 0$, and hence J is weakly n -absorbing primary. Assume that $J^{n+1} \neq 0$. Put $R_2 = J^{n+1} = Re$ and $R_1 = R(1 - e)$; hence $R = R_1 \times R_2$. Let $I = J(1 - e)$, so $J = I \times R_2$, where $I^{n+1} = 0$. We show that I is weakly n -absorbing primary. Let $x_1, x_2, \dots, x_{n+1} \in R$ and $x_1 x_2 \cdots x_{n+1} \in I \setminus \{0\}$ such that $x_1 x_2 \cdots x_n \notin I$. So $(x_1, 0)(x_2, 0) \cdots (x_{n+1}, 0) = (x_1 x_2 \cdots x_{n+1}, 0) \in I \times R_2 = J$. Since $J^{n+1} = \{0\} \times R_2$ and $\phi(J) \subseteq J^{n+1}$, then $(x_1, 0)(x_2, 0) \cdots (x_{n+1}, 0) = (x_1 x_2 \cdots x_{n+1}, 0) \in J \setminus \phi(J)$. Since J is ϕ - n -absorbing primary, so either $(x_1, 0)(x_2, 0) \cdots (x_n, 0) = (x_1 x_2 \cdots x_n, 0) \in I \times R_2 = J$ or $(x_1, 0) \cdots \widehat{(x_i, 0)} \cdots (x_{n+1}, 0) = (x_1 \cdots \widehat{x_i} \cdots x_{n+1}, 0) \in \sqrt{I} \times R_2 = \sqrt{J}$ for some $1 \leq i \leq n$. The first case implies that $x_1 x_2 \cdots x_n \in I$, which is a contradiction. The second case implies that $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Consequently I is weakly n -absorbing primary. \square

Corollary 3.3. *Let R be an indecomposable ring and J a finitely generated ϕ - n -absorbing primary ideal of R , where $\phi \leq \phi_{n+2}$. Then J is weakly n -absorbing primary. Furthermore, if R is an integral domain, then J is actually n -absorbing primary.*

Corollary 3.4. *Let R be a Noetherian integral domain. A proper ideal J of R is n -absorbing primary if and only if it is $(n + 2)$ -almost n -absorbing primary.*

Theorem 3.5. *Let $R = R_1 \times \cdots \times R_s$ be a decomposable ring and $\psi_i : \mathfrak{J}(R_i) \rightarrow \mathfrak{J}(R_i) \cup \{\emptyset\}$ be a function for $i = 1, 2, \dots, s$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_n$. Suppose that*

$$L = I_1 \times \cdots \times I_{\alpha_1-1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times R_{\alpha_j} \times I_{\alpha_j+1} \times \cdots \times I_s$$

be an ideal of R in which $\{\alpha_1, \dots, \alpha_j\} \subset \{1, \dots, s\}$. Moreover, suppose that $\psi_{\alpha_i}(R_{\alpha_i}) \neq R_{\alpha_i}$ for some $\alpha_i \in \{\alpha_1, \dots, \alpha_j\}$. The following conditions are equivalent:

- (1) L is a ϕ - n -absorbing ideal of R ;
- (2) L is an n -absorbing primary ideal of R ;
- (3) $L' := I_1 \times \cdots \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times I_{\alpha_j+1} \times \cdots \times I_s$ is an n -absorbing primary ideal of

$$R' := R_1 \times \cdots \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_j-1} \times R_{\alpha_j+1} \times \cdots \times R_s.$$

Proof. (1) \Rightarrow (2) Since $\psi_{\alpha_i}(R_{\alpha_i}) \neq R_{\alpha_i}$ for some $\alpha_i \in \{\alpha_1, \dots, \alpha_j\}$, then clearly $L \not\subseteq \sqrt{\phi(L)}$. So by Theorem 2.30(2), L is an n -absorbing ideal of R .

(2) \Rightarrow (3) Assume that L is an n -absorbing primary ideal of R and

$$\begin{aligned} (a_1^{(1)}, \dots, a_{\alpha_1-1}^{(1)}, & \quad a_{\alpha_1+1}^{(1)}, \dots, a_{\alpha_j-1}^{(1)}, a_{\alpha_j+1}^{(1)}, \dots, a_s^{(1)}) \dots \\ & (a_1^{(n+1)}, \dots, a_{\alpha_1-1}^{(n+1)}, a_{\alpha_1+1}^{(n+1)}, \dots, a_{\alpha_j-1}^{(n+1)}, a_{\alpha_j+1}^{(n+1)}, \dots, a_s^{(n+1)}) \in L', \end{aligned}$$

in which $a_i^{(t)}$'s are in R_i , respectively. Then

$$\begin{aligned} (a_1^{(1)}, \dots, a_{\alpha_1-1}^{(1)}, & \quad 1, a_{\alpha_1+1}^{(1)}, \dots, a_{\alpha_j-1}^{(1)}, 1, a_{\alpha_j+1}^{(1)}, \dots, a_s^{(1)}) \dots \\ & (a_1^{(n+1)}, \dots, a_{\alpha_1-1}^{(n+1)}, 1, a_{\alpha_1+1}^{(n+1)}, \dots, a_{\alpha_j-1}^{(n+1)}, 1, a_{\alpha_j+1}^{(n+1)}, \dots, a_s^{(n+1)}) \in L. \end{aligned}$$

So, either

$$\begin{aligned} (a_1^{(1)}, \dots, a_{\alpha_1-1}^{(1)}, & \quad 1, a_{\alpha_1+1}^{(1)}, \dots, a_{\alpha_j-1}^{(1)}, 1, a_{\alpha_j+1}^{(1)}, \dots, a_s^{(1)}) \dots \\ & (a_1^{(n)}, \dots, a_{\alpha_1-1}^{(n)}, 1, a_{\alpha_1+1}^{(n)}, \dots, a_{\alpha_j-1}^{(n)}, 1, a_{\alpha_j+1}^{(n)}, \dots, a_s^{(n)}) \in L. \end{aligned}$$

or, there exists $1 \leq i \leq n$ such that

$$\begin{aligned} (a_1^{(1)}, \dots, a_{\alpha_1-1}^{(1)}, & \quad 1, a_{\alpha_1+1}^{(1)}, \dots, a_{\alpha_j-1}^{(1)}, 1, a_{\alpha_j+1}^{(1)}, \dots, a_s^{(1)}) \dots \\ & (a_1^{(i-1)}, \dots, a_{\alpha_1-1}^{(i-1)}, 1, a_{\alpha_1+1}^{(i-1)}, \dots, a_{\alpha_j-1}^{(i-1)}, 1, a_{\alpha_j+1}^{(i-1)}, \dots, a_s^{(i-1)}) \\ & (a_1^{(i+1)}, \dots, a_{\alpha_1-1}^{(i+1)}, 1, a_{\alpha_1+1}^{(i+1)}, \dots, a_{\alpha_j-1}^{(i+1)}, 1, a_{\alpha_j+1}^{(i+1)}, \dots, a_s^{(i+1)}) \dots \\ & (a_1^{(n+1)}, \dots, a_{\alpha_1-1}^{(n+1)}, 1, a_{\alpha_1+1}^{(n+1)}, \dots, a_{\alpha_j-1}^{(n+1)}, 1, a_{\alpha_j+1}^{(n+1)}, \dots, a_s^{(n+1)}) \in \sqrt{L}, \end{aligned}$$

because L is an n -absorbing primary ideal of R . Hence, either

$$\begin{aligned} (a_1^{(1)}, \dots, a_{\alpha_1-1}^{(1)}, & \quad a_{\alpha_1+1}^{(1)}, \dots, a_{\alpha_j-1}^{(1)}, a_{\alpha_j+1}^{(1)}, \dots, a_s^{(1)}) \dots \\ & (a_1^{(n)}, \dots, a_{\alpha_1-1}^{(n)}, a_{\alpha_1+1}^{(n)}, \dots, a_{\alpha_j-1}^{(n)}, a_{\alpha_j+1}^{(n)}, \dots, a_s^{(n)}) \in L'. \end{aligned}$$

or there exists $1 \leq i \leq n$ such that

$$\begin{aligned} (a_1^{(1)}, \dots, a_{\alpha_1-1}^{(1)}, & \quad a_{\alpha_1+1}^{(1)}, \dots, a_{\alpha_j-1}^{(1)}, a_{\alpha_j+1}^{(1)}, \dots, a_s^{(1)}) \dots \\ & (a_1^{(i-1)}, \dots, a_{\alpha_1-1}^{(i-1)}, a_{\alpha_1+1}^{(i-1)}, \dots, a_{\alpha_j-1}^{(i-1)}, a_{\alpha_j+1}^{(i-1)}, \dots, a_s^{(i-1)}) \\ & (a_1^{(i+1)}, \dots, a_{\alpha_1-1}^{(i+1)}, a_{\alpha_1+1}^{(i+1)}, \dots, a_{\alpha_j-1}^{(i+1)}, a_{\alpha_j+1}^{(i+1)}, \dots, a_s^{(i+1)}) \dots \\ & (a_1^{(n+1)}, \dots, a_{\alpha_1-1}^{(n+1)}, a_{\alpha_1+1}^{(n+1)}, \dots, a_{\alpha_j-1}^{(n+1)}, a_{\alpha_j+1}^{(n+1)}, \dots, a_s^{(n+1)}) \in \sqrt{L'}, \end{aligned}$$

Consequently, L' is an n -absorbing primary ideal of R' .

(3) \Rightarrow (1) Let L' is an n -absorbing primary ideal of R' . It is routine to see that L is an n -absorbing primary ideal of R . Consequently, L is a ϕ - n -absorbing primary ideal of R . \square

Theorem 3.6. Let $R = R_1 \times \dots \times R_n$ be a ring with identity and let $\psi_i : \mathfrak{I}(R_i) \rightarrow \mathfrak{I}(R_i) \cup \{\emptyset\}$ be a function for $i = 1, 2, \dots, n$ such that $\psi_n(R_n) \neq R_n$. Set $\phi = \psi_1 \times \psi_2 \times \dots \times \psi_n$. Suppose that $I_1 \times I_2 \times \dots \times I_n$ is an ideal of R which $\psi_1(I_1) \neq I_1$, and for some $2 \leq j \leq n$, $\psi_j(I_j) \neq I_j$, and I_i is a proper ideal of R_i for each $1 \leq i \leq n-1$. The following conditions are equivalent:

- (1) $I_1 \times I_2 \times \dots \times I_n$ is a ϕ - n -absorbing primary ideal of R ;
- (2) $I_n = R_n$ and $I_1 \times I_2 \times \dots \times I_{n-1}$ is an n -absorbing primary ideal of $R_1 \times \dots \times R_{n-1}$ or I_n is a primary ideal of R_n and for every $1 \leq i \leq n-1$, I_i is a primary ideal of R_i , respectively;
- (3) $I_1 \times I_2 \times \dots \times I_n$ is an n -absorbing primary ideal of R .

Proof. (1) \Rightarrow (2) Suppose that $I_1 \times I_2 \times \cdots \times I_n$ is a ϕ - n -absorbing primary ideal of R . First assume that $I_n = R_n$. Since $\psi_n(R_n) \neq R_n$, then $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an n -absorbing primary ideal of $R_1 \times \cdots \times R_{n-1}$, by Theorem 3.5. Now, suppose that $I_n \neq R_n$. Fix $2 \leq i \leq n$. We show that I_i is a primary ideal of R_i . Suppose that $ab \in I_i$ for some $a, b \in R_i$. Let $x \in I_1 \setminus \psi_1(I_1)$. Then

$$\begin{aligned} & (x, 1, \dots, 1)(1, 0, 1, \dots, 1, \dots, 1)(1, 1, 0, 1, \dots, 1, \dots, 1) \cdots \\ & (1, \dots, 1, 0, \overbrace{1}^{i-th}, \dots, 1)(1, \dots, \overbrace{1}^{i-th}, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0) \\ & (1, \dots, 1, \overbrace{a}^{i-th}, 1, \dots, 1)(1, \dots, 1, \overbrace{b}^{i-th}, 1, \dots, 1) \\ & = (x, 0, \dots, 0, \overbrace{ab}^{i-th}, 0, \dots, 0) \in I_1 \times \cdots \times I_n \setminus \psi_1(I_1) \times \cdots \times \psi_n(I_n), \end{aligned}$$

Since $I_1 \times I_2 \times \cdots \times I_n$ is ϕ - n -absorbing primary and I_i 's are proper, then either

$$\begin{aligned} & (x, 1, \dots, 1)(1, 0, 1, \dots, 1, \dots, 1)(1, 1, 0, 1, \dots, 1, \dots, 1) \cdots \\ & (1, \dots, 1, 0, \overbrace{1}^{i-th}, \dots, 1)(1, \dots, \overbrace{1}^{i-th}, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0) \\ & (1, \dots, 1, \overbrace{a}^{i-th}, 1, \dots, 1) = (x, 0, \dots, 0, \overbrace{a}^{i-th}, 0, \dots, 0) \in I_1 \times \cdots \times I_n, \end{aligned}$$

or

$$\begin{aligned} & (x, 1, \dots, 1)(1, 0, 1, \dots, 1, \dots, 1)(1, 1, 0, 1, \dots, 1, \dots, 1) \cdots \\ & (1, \dots, 1, 0, \overbrace{1}^{i-th}, \dots, 1)(1, \dots, \overbrace{1}^{i-th}, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0) \\ & (1, \dots, 1, \overbrace{b}^{i-th}, 1, \dots, 1) = (x, 0, \dots, 0, \overbrace{b}^{i-th}, 0, \dots, 0) \in \sqrt{I_1 \times \cdots \times I_n}, \end{aligned}$$

and thus either $a \in I_i$ or $b \in \sqrt{I_i}$. Consequently I_i is a primary ideal of R_i . Since for some $2 \leq j \leq n$, $\psi_j(I_j) \neq I_j$, similarly we can show that I_1 is a primary ideal of R_1 .

(2) \Rightarrow (3) If $I_n = R_n$ and $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an n -absorbing primary ideal of $R_1 \times \cdots \times R_{n-1}$, then $I_1 \times I_2 \times \cdots \times I_n$ is an n -absorbing primary ideal of R , by Theorem 3.5. Now, assume that I_n is a primary ideal of R_n and for each $1 \leq i \leq n-1$, I_i is a primary ideal of R_i . Suppose that

$$\begin{aligned} & (a_1^{(1)}, \dots, a_n^{(1)})(a_1^{(2)}, \dots, a_n^{(2)}) \cdots (a_1^{(n+1)}, \dots, a_n^{(n+1)}) \\ & \in I_1 \times I_2 \times \cdots \times I_n \setminus \psi_1(I_1) \times \cdots \times \psi_n(I_n), \end{aligned}$$

in which for every $1 \leq j \leq n+1$, $a_i^{(j)}$'s are in R_i , respectively. Suppose that

$$(a_1^{(1)}, \dots, a_n^{(1)})(a_1^{(2)}, \dots, a_n^{(2)}) \cdots (a_1^{(n)}, \dots, a_n^{(n)}) \notin I_1 \times I_2 \times \cdots \times I_n.$$

Without loss of generality we may assume that $a_1^{(1)} \cdots a_n^{(n)} \notin I_1$. Since I_1 is primary, we deduce that $a_1^{(n+1)} \in \sqrt{I_1}$. On the other hand $\sqrt{I_i}$ is a prime ideal, for any $2 \leq i \leq n$, then at least one of the $a_i^{(j)}$'s is in $\sqrt{I_i}$, say $a_i^{(i)} \in \sqrt{I_i}$. Thus $(a_1^{(2)}, \dots, a_n^{(2)}) \cdots (a_1^{(n+1)}, \dots, a_n^{(n+1)}) \in \sqrt{I_1 \times I_2 \times \cdots \times I_n}$. Consequently $I_1 \times I_2 \times \cdots \times I_n$ is an n -absorbing primary ideal of R .

(3) \Rightarrow (1) is obvious. \square

Theorem 3.7. Let $R = R_1 \times \cdots \times R_n$ be a ring with identity and let $\psi_i : \mathfrak{I}(R_i) \rightarrow \mathfrak{I}(R_i) \cup \{\emptyset\}$ be a function for $i = 1, 2, \dots, n$ such that $\psi_n(R_n) \neq R_n$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_n$, and suppose that for every $1 \leq i \leq n-1$, I_i is a proper ideal of R_i such that $\psi_1(I_1) \neq I_1$ and I_n is an ideal of R_n . Consider the following conditions:

- (1) $I_1 \times \cdots \times I_n$ is a ϕ - n -absorbing primary ideal of R that is not an n -absorbing primary ideal of R .
- (2) I_1 is a ψ_1 -primary ideal of R_1 that is not a primary ideal and for every $2 \leq i \leq n$, $I_i = \psi_i(I_i)$ is a primary ideal of R_i , respectively.

Then (1) \Rightarrow (2).

Proof. (1) \Rightarrow (2) Assume that $I_1 \times \cdots \times I_n$ is a ϕ - n -absorbing primary ideal of R that is not an n -absorbing primary ideal. If for some $2 \leq i \leq n$ we have $\psi_i(I_i) \neq I_i$, then $I_1 \times \cdots \times I_n$ is an n -absorbing primary ideal of R by Theorem 3.6, which contradicts our assumption. Thus for every $2 \leq i \leq n$, $\psi_i(I_i) = I_i$ and so $I_n \neq R_n$. A proof similar to part (1) \Rightarrow (2) of Theorem 3.6 shows that for every $2 \leq i \leq n$, $\psi_i(I_i) = I_i$ is a primary ideal of R_i . Now, we show that I_1 is a ψ_1 -primary ideal of R_1 . Consider $a, b \in R_1$ such that $ab \in I_1 \setminus \psi_1(I_1)$. Note that

$$\begin{aligned} & (1, 0, 1, \dots, 1)(1, 1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0)(a, 1, \dots, 1)(b, 1, \dots, 1) \\ &= (ab, 0, \dots, 0) \in (I_1 \times I_2 \times \cdots \times I_n) \setminus (\psi_1(I_1) \times \cdots \times \psi_n(I_n)). \end{aligned}$$

Because I_i 's are proper, the product of $(a, 1, \dots, 1)(b, 1, \dots, 1)$ with $n-2$ of $(1, 0, 1, \dots, 1), (1, 1, 0, 1, \dots, 1), \dots, (1, \dots, 1, 0)$ is not in $\sqrt{I_1 \times I_2 \times \cdots \times I_n}$. Since $I_1 \times I_2 \times \cdots \times I_n$ is a ϕ - n -absorbing primary ideal of R , we have either

$$\begin{aligned} & (1, 0, 1, \dots, 1)(1, 1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0)(a, 1, \dots, 1) \\ &= (a, 0, \dots, 0) \in I_1 \times I_2 \times \cdots \times I_n. \end{aligned}$$

or

$$\begin{aligned} & (1, 0, 1, \dots, 1)(1, 1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0)(b, 1, \dots, 1) \\ &= (b, 0, \dots, 0) \in \sqrt{I_1 \times I_2 \times \cdots \times I_n}. \end{aligned}$$

So either $a \in I_1$ or $b \in \sqrt{I_1}$. Thus I_1 is a ψ_1 -primary ideal of R_1 . Assume I_1 is a primary ideal of R_1 , since for every $2 \leq i \leq n$, I_i is a primary ideal of R_i , it is easy to see that $I_1 \times \cdots \times I_n$ is an n -absorbing primary ideal of R , which is a contradiction. \square

Theorem 3.8. Let $R = R_1 \times R_2 \times R_3$ be a ring with identity and let $\psi_i : \mathfrak{I}(R_i) \rightarrow \mathfrak{I}(R_i) \cup \{\emptyset\}$ be a function for $i = 1, 2, 3$ such that $\psi_3(R_3) \neq R_3$. Set $\phi = \psi_1 \times \psi_2 \times \psi_3$, and suppose that for $i = 1, 2$, I_i is a proper ideal of R_i such that $\psi_1(I_1) \neq I_1$ and I_3 is an ideal of R_3 . The following conditions are equivalent:

- (1) $I_1 \times I_2 \times I_3$ is a ϕ -3-absorbing primary ideal of R that is not an 3-absorbing primary ideal of R .
- (2) I_1 is a ψ_1 -primary ideal of R_1 that is not a primary ideal and for $i = 2, 3$, $I_i = \psi_i(I_i)$ is a primary ideal of R_i , respectively.

Proof. (1) \Rightarrow (2) See Theorem 3.7.

(2) \Rightarrow (1) Let

$$\begin{aligned} & (a_1^{(1)}, a_2^{(1)}, a_3^{(1)})(a_1^{(2)}, a_2^{(2)}, a_3^{(2)})(a_1^{(3)}, a_2^{(3)}, a_3^{(3)})(a_1^{(4)}, a_2^{(4)}, a_3^{(4)}) \\ & \in I_1 \times \psi_2(I_2) \times \psi_3(I_3) \setminus \psi_1(I_1) \times \psi_2(I_2) \times \psi_3(I_3), \end{aligned}$$

in which, for every $1 \leq i \leq 3$, $a_i^{(1)}, a_i^{(2)}, a_i^{(3)}, a_i^{(4)} \in R_i$, respectively. Assume that

$$(a_1^{(1)}, a_2^{(1)}, a_3^{(1)})(a_1^{(2)}, a_2^{(2)}, a_3^{(2)})(a_1^{(3)}, a_2^{(3)}, a_3^{(3)}) \notin I_1 \times \psi_2(I_2) \times \psi_3(I_3).$$

Therefore, we may have three cases; $a_1^{(1)}a_1^{(2)}a_1^{(3)} \notin I_1$ or $a_2^{(1)}a_2^{(2)}a_2^{(3)} \notin I_2$ or $a_3^{(1)}a_3^{(2)}a_3^{(3)} \notin I_3$.

Case 1. Assume that $a_1^{(1)}a_1^{(2)}a_1^{(3)} \notin I_1$. Since $a_1^{(1)}a_1^{(2)}a_1^{(3)}a_1^{(4)} \in I_1 \setminus \psi_1(I_1)$ and I_1 is ψ_1 -primary, then $a_1^{(4)} \in \sqrt{I_1}$. On the other hand $\sqrt{I_2}$ and $\sqrt{I_3}$ are prime and $a_2^{(1)}a_2^{(2)}a_2^{(3)}a_2^{(4)} \in \sqrt{I_2}$ and $a_3^{(1)}a_3^{(2)}a_3^{(3)}a_3^{(4)} \in \sqrt{I_3}$. Without loss of generality we may assume that $a_2^{(2)} \in \sqrt{I_2}$ and $a_3^{(3)} \in \sqrt{I_3}$. Hence

$$(a_1^{(2)}, a_2^{(2)}, a_3^{(2)})(a_1^{(3)}, a_2^{(3)}, a_3^{(3)})(a_1^{(4)}, a_2^{(4)}, a_3^{(4)}) \in \sqrt{I_1 \times \psi_2(I_2) \times \psi_3(I_3)}.$$

Case 2. Assume that $a_2^{(1)}a_2^{(2)}a_2^{(3)} \notin I_2$. Since I_2 is primary and $a_2^{(1)}a_2^{(2)}a_2^{(3)}a_2^{(4)} \in I_2$, then $a_2^{(4)} \in \sqrt{I_2}$. Since $\sqrt{I_3}$ is prime and $a_3^{(1)}a_3^{(2)}a_3^{(3)}a_3^{(4)} \in \sqrt{I_3}$, then at least one of the $a_3^{(j)}$'s is in $\sqrt{I_3}$. Let $a_3^{(1)} \in \sqrt{I_3}$. Since I_1 is ψ_1 -primary and $a_1^{(1)}a_1^{(2)}a_1^{(3)} \in I_1 \setminus \psi_1(I_1)$, then either $a_1^{(3)} \in I_1$ or $a_1^{(1)}a_1^{(2)} \in \sqrt{I_1}$. So

$$(a_1^{(1)}, a_2^{(1)}, a_3^{(1)})(a_1^{(3)}, a_2^{(3)}, a_3^{(3)})(a_1^{(4)}, a_2^{(4)}, a_3^{(4)}) \in \sqrt{I_1 \times \psi_2(I_2) \times \psi_3(I_3)}$$

or

$$(a_1^{(1)}, a_2^{(1)}, a_3^{(1)})(a_1^{(2)}, a_2^{(2)}, a_3^{(2)})(a_1^{(4)}, a_2^{(4)}, a_3^{(4)}) \in \sqrt{I_1 \times \psi_2(I_2) \times \psi_3(I_3)}.$$

Let $a_3^{(2)} \in \sqrt{I_3}$. Since I_1 is ψ_1 -primary, we have either $a_1^{(1)} \in I_1$ or $a_1^{(2)}a_1^{(3)} \in \sqrt{I_1}$. Hence

$$(a_1^{(1)}, a_2^{(1)}, a_3^{(1)})(a_1^{(2)}, a_2^{(2)}, a_3^{(2)})(a_1^{(4)}, a_2^{(4)}, a_3^{(4)}) \in \sqrt{I_1 \times \psi_2(I_2) \times \psi_3(I_3)}$$

or

$$(a_1^{(2)}, a_2^{(2)}, a_3^{(2)})(a_1^{(3)}, a_2^{(3)}, a_3^{(3)})(a_1^{(4)}, a_2^{(4)}, a_3^{(4)}) \in \sqrt{I_1 \times \psi_2(I_2) \times \psi_3(I_3)}.$$

With a similar manner we can investigate the other remaining situations.

Case 3. For $a_3^{(1)}a_3^{(2)}a_3^{(3)} \notin I_3$ we act like case 2. Consequently $I_1 \times I_2 \times I_3$ is a ϕ -3-absorbing primary ideal of R . Since I_1 is not a primary ideal of R_1 , there exist elements $a, b \in R_1$ such that $ab \in \psi_1(I_1)$, but $a \notin I_1$ and $b \notin \sqrt{I_1}$. Hence

$$(1, 0, 1)(1, 1, 0)(a, 1, 1)(b, 1, 1) = (ab, 0, 0) \in \psi_1(I_1) \times \psi_2(I_2) \times \psi_3(I_3),$$

but neither

$$(1, 0, 1)(1, 1, 0)(a, 1, 1) = (a, 0, 0) \in I_1 \times I_2 \times I_3,$$

nor

$$(1, 0, 1)(1, 1, 0)(b, 1, 1) = (b, 0, 0) \in \sqrt{I_1 \times I_2 \times I_3},$$

also the product of $(a, 1, 1)(b, 1, 1)$ with any one of elements $(1, 0, 1), (1, 1, 0)$ is not in $\sqrt{I_1 \times I_2 \times I_3}$. Consequently $I_1 \times I_2 \times I_3$ is not a 3-absorbing primary ideal of R . □

Theorem 3.9. Let $R = R_1 \times \cdots \times R_{n+1}$ where R_i 's are rings with identity and let for $i = 1, 2, \dots, n+1$, $\psi_i : \mathfrak{I}(R_i) \rightarrow \mathfrak{I}(R_i) \cup \{\emptyset\}$ be a function such that $\psi_i(R_i) \neq R_i$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1}$.

- (1) For every ideal I of R , $\phi(I)$ is not an n -absorbing primary ideal of R ;
- (2) If I is a ϕ - n -absorbing primary ideal of R , then either $I = \phi(I)$, or I is an n -absorbing primary ideal of R .

Proof. Let I be an ideal of R . We know that the ideal I is of the form $I_1 \times \cdots \times I_{n+1}$ where I_i 's are ideals of R_i 's, respectively.

(1) Suppose that $\phi(I)$ is an n -absorbing primary ideal of R . Since

$$\begin{aligned} (0, 1, \dots, 1)(1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0) &= (0, \dots, 0) \\ &\in \phi(I) = \psi_1(I_1) \times \cdots \times \psi_{n+1}(I_{n+1}), \end{aligned}$$

we have that either

$$\begin{aligned} (0, 1, \dots, 1)(1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0, 1) &= (0, \dots, 0, 1) \\ &\in \psi_1(I_1) \times \cdots \times \psi_{n+1}(I_{n+1}), \end{aligned}$$

or the product of $(1, \dots, 1, 0)$ with $n-1$ of $(0, 1, \dots, 1), (1, 0, 1, \dots, 1), \dots, (1, \dots, 1, 0, 1)$ is in $\sqrt{\phi(I)}$. Hence, for some $1 \leq i \leq n+1$, $1 \in \psi_i(I_i)$ which implies that $\psi_i(R_i) = R_i$, a contradiction. Consequently $\phi(I)$ is not an n -absorbing primary ideal of R .

(2) If $I = \phi(I)$, then clearly I is a ϕ - n -absorbing primary ideal of R , we may assume that $I = I_1 \times \cdots \times I_{n+1} \neq \psi_1(I_1) \times \psi_2(I_2) \times \cdots \times \psi_{n+1}(I_{n+1})$. So, there is an element $(a_1, \dots, a_{n+1}) \in I \setminus (\psi_1(I_1) \times \psi_2(I_2) \times \cdots \times \psi_{n+1}(I_{n+1}))$. Then $(a_1, 1, \dots, 1)(1, a_2, 1, \dots, 1) \cdots (1, \dots, 1, a_{n+1}) \in I$. Since I is a ϕ - n -absorbing primary ideal of R , then either

$$\begin{aligned} (a_1, 1, \dots, 1)(1, a_2, 1, \dots, 1) \cdots (1, \dots, 1, a_n, 1) \\ = (a_1, a_2, \dots, a_n, 1) \in I. \end{aligned}$$

or, for some $1 \leq i \leq n$ we have

$$\begin{aligned} (a_1, 1, \dots, 1) \cdots (1, \dots, 1, a_{i-1}, 1, \dots, 1)(1, \dots, 1, a_{i+1}, 1, \dots, 1) \cdots \\ (1, \dots, 1, a_{n+1}) = (a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_{n+1}) \in \sqrt{I}. \end{aligned}$$

Then $I_i = R_i$, for some $1 \leq i \leq n+1$ and so $I = I_1 \times \cdots \times I_{i-1} \times R_i \times I_{i+1} \times \cdots \times I_{n+1}$. If $I \subseteq \sqrt{\phi(I)}$, then $\psi_i(R_i) = R_i$ which is a contradiction. Therefore, by Theorem 2.30, I must be an n -absorbing primary ideal of R . \square

Theorem 3.10. Let $R = R_1 \times \cdots \times R_{n+1}$ where R_i 's are rings with identity and let for $i = 1, 2, \dots, n+1$, $\psi_i : \mathfrak{I}(R_i) \rightarrow \mathfrak{I}(R_i) \cup \{\emptyset\}$ be a function such that $\psi_i(R_i) \neq R_i$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1}$. Let $L = I_1 \times \cdots \times I_{n+1}$ be a proper ideal of R with $L \neq \phi(L)$. The following conditions are equivalent:

- (1) $L = I_1 \times \cdots \times I_{n+1}$ is a ϕ - n -absorbing primary ideal of R ;
- (2) $L = I_1 \times \cdots \times I_{n+1}$ is an n -absorbing primary ideal of R ;
- (3) $L = I_1 \times \cdots \times I_{i-1} \times R_i \times I_{i+1} \times \cdots \times I_{n+1}$ for some $1 \leq i \leq n+1$ such that for each $1 \leq t \leq n+1$ different from i , I_t is a primary ideal of R_t or $L = I_1 \times \cdots \times I_{\alpha_1-1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times R_{\alpha_j} \times I_{\alpha_j+1} \cdots \times I_{n+1}$ in which $\{\alpha_1, \dots, \alpha_j\} \subsetneq \{1, \dots, n+1\}$ and

$$I_1 \times \cdots \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times I_{\alpha_j+1} \cdots \times I_{n+1}$$

is an n -absorbing primary ideal of

$$R_1 \times \cdots \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_j-1} \times R_{\alpha_j+1} \times \cdots \times R_{n+1}.$$

Proof. (1) \Rightarrow (2) Since L is a ϕ - n -absorbing primary ideal of R and $L \neq \phi(L)$, then L is an n -absorbing primary ideal of R , by Theorem 3.9.

(2) \Rightarrow (3) Suppose that L is an n -absorbing primary ideal of R , then for some $1 \leq i \leq n+1$, $I_i = R_i$ by the proof of Theorem 3.9. Assume that $L = I_1 \times \cdots \times I_{i-1} \times R_i \times I_{i+1} \times \cdots \times I_{n+1}$ for $1 \leq i \leq n+1$ such that for each $1 \leq t \leq n+1$ different

from i , I_t is a proper ideal of R_t . Fix an I_t different from I_i . We may assume that $t > i$. Let $ab \in I_t$ for some $a, b \in R_t$. In this case

$$\begin{aligned} & (0, 1, \dots, 1)(1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0, \overbrace{1}^{i-th}, \dots, 1)(1, \dots, \overbrace{1}^{i-th}, 0, 1, \dots, 1) \cdots \\ & (1, \dots, 1, 0, \overbrace{1}^{t-th}, \dots, 1)(1, \dots, \overbrace{1}^{t-th}, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0)(1, \dots, 1, \overbrace{a}^{t-th}, 1, \dots, 1) \\ & (1, \dots, 1, \overbrace{b}^{t-th}, 1, \dots, 1) = (0, \dots, 0, \overbrace{1}^{i-th}, 0, \dots, 0, \overbrace{ab}^{t-th}, 0, \dots, 0) \in L. \end{aligned}$$

Since $I_1 \times \cdots \times I_{n+1}$ is n -absorbing primary and I_j 's different from I_i are proper, then either

$$\begin{aligned} & (0, 1, \dots, 1)(1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0, \overbrace{1}^{i-th}, \dots, 1)(1, \dots, \overbrace{1}^{i-th}, 0, 1, \dots, 1) \cdots \\ & (1, \dots, 1, 0, \overbrace{1}^{t-th}, \dots, 1)(1, \dots, \overbrace{1}^{t-th}, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0)(1, \dots, 1, \overbrace{a}^{t-th}, 1, \dots, 1) \\ & = (0, \dots, 0, \overbrace{1}^{i-th}, 0, \dots, 0, \overbrace{a}^{t-th}, 0, \dots, 0) \in L, \end{aligned}$$

or

$$\begin{aligned} & (0, 1, \dots, 1)(1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0, \overbrace{1}^{i-th}, \dots, 1)(1, \dots, \overbrace{1}^{i-th}, 0, 1, \dots, 1) \cdots \\ & (1, \dots, 1, 0, \overbrace{1}^{t-th}, \dots, 1)(1, \dots, \overbrace{1}^{t-th}, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0)(1, \dots, 1, \overbrace{b}^{t-th}, 1, \dots, 1) \\ & = (0, \dots, 0, \overbrace{1}^{i-th}, 0, \dots, 0, \overbrace{b}^{t-th}, 0, \dots, 0) \in \sqrt{L}, \end{aligned}$$

and thus either $a \in I_t$ or $b \in \sqrt{I_t}$. Consequently I_t is a primary ideal of R_t .

Now, assume that

$$L = I_1 \times \cdots \times I_{\alpha_1-1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times R_{\alpha_j} \times I_{\alpha_j+1} \times \cdots \times I_{n+1}$$

in which $\{\alpha_1, \dots, \alpha_j\} \subsetneq \{1, \dots, n+1\}$. Since L is n -absorbing primary, then $I_1 \times \cdots \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times I_{\alpha_j+1} \times \cdots \times I_{n+1}$ is an n -absorbing primary ideal of

$$R_1 \times \cdots \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_j-1} \times R_{\alpha_j+1} \times \cdots \times R_{n+1},$$

by Theorem 3.5.

(3) \Rightarrow (1) If L is in the first form, then similar to the proof of part (2) \Rightarrow (3) of Theorem 3.6 we can verify that L is an n -absorbing primary ideal of R , and hence L is a ϕ - n -absorbing primary ideal of R . For the second form apply Theorem 3.5. \square

Theorem 3.11. Let $R = R_1 \times \cdots \times R_{n+1}$ where R_i 's are rings with identity and let for $i = 1, 2, \dots, n+1$, $\psi_i : \mathfrak{I}(R_i) \rightarrow \mathfrak{I}(R_i) \cup \{\emptyset\}$ be a function. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1}$. If every proper ideal of R is a ϕ - n -absorbing primary ideal (ϕ - n -absorbing ideal) of R , then $I = \psi_i(I)$ for every $1 \leq i \leq n+1$ and every proper ideal I of R_i . The converse holds if in addition $\psi_i(R_i) = R_i$ for every $1 \leq i \leq n+1$.

Proof. Fix an i and let I be a proper ideal of R_i . Assume that $I \neq \psi_i(I)$, so give an element $x \in I \setminus \psi_i(I)$. Set

$$J := I \times \{0\} \cdots \times \{0\}.$$

Notice that

$$(1, 0, 1, \dots, 1)(1, 1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0)(x, 1, \dots, 1) \in J \setminus \phi(J)$$

Since I is ϕ - n -absorbing primary, then either

$$(1, 0, 1, \dots, 1)(1, 1, 0, 1, \dots, 1) \cdots (1, \dots, 1, 0) \in J,$$

or the product of $(x, 1, \dots, 1)$ with $n - 1$ of $(1, 0, 1, \dots, 1), (1, 1, 0, 1, \dots, 1), \dots, (1, \dots, 1, 0)$ is in \sqrt{J} which implies that either $1 \in I$ or $1 \in \{0\}$, a contradiction. Consequently $I = \psi_i(I)$. \square

Corollary 3.12. *Let $n \geq 2$ be a natural number and $R = R_1 \times \cdots \times R_{n+1}$ be a decomposable ring with identity. The following conditions are equivalent:*

- (1) R is a von Neumann regular ring;
- (2) Every proper ideal of R is an n -almost n -absorbing primary ideal of R ;
- (3) Every proper ideal of R is an ω - n -absorbing primary ideal of R ;
- (4) Every proper ideal of R is an n -almost n -absorbing ideal of R .

Proof. (1) \Leftrightarrow (2), (1) \Leftrightarrow (3) and (1) \Leftrightarrow (4): Notice that, $\phi_n(I) = I$ (or $\phi_\omega(I) = I$) if and only if $I = I^2$. By the fact that R is von Neumann regular if and only if $I = I^2$ for every ideal I of R and regarding Theorem 3.11 we have the implications. \square

Corollary 3.13. *Let R_1, R_2, \dots, R_{n+1} be rings and let $R = R_1 \times R_2 \times \cdots \times R_{n+1}$. Then the following conditions are equivalent:*

- (1) R_1, R_2, \dots, R_{n+1} are fields;
- (2) Every proper ideal of R is a weakly n -absorbing ideal of R ;
- (3) Every proper ideal of R is a weakly n -absorbing primary ideal of R .

Proof. (1) \Rightarrow (2) By [11, Theorem 1.10].

(2) \Rightarrow (3) is clear.

(3) \Rightarrow (1) In Theorem 3.11 assume that $\phi = 0$. \square

4. THE STABILITY OF ϕ - n -ABSORBING PRIMARY IDEALS WITH RESPECT TO IDEALIZATION

Let R be a commutative ring and M be an R -module. We recall from [14, Theorem 25.1] that every ideal of $R(+)M$ is in the form of $I(+)N$ in which I is an ideal of R and N is a submodule of M such that $IM \subseteq N$. Moreover, if $I_1(+)N_1$ and $I_2(+)N_2$ are ideals of $R(+)M$, then $(I_1(+)N_1) \cap (I_2(+)N_2) = (I_1 \cap I_2)(+)(N_1 \cap N_2)$.

Theorem 4.1. *Let R be a ring, I a proper ideal of R and M an R -module. Suppose that $\psi : \mathfrak{I}(R) \rightarrow \mathfrak{I}(R) \cup \{\emptyset\}$ and $\phi : \mathfrak{I}(R(+)M) \rightarrow \mathfrak{I}(R(+)M) \cup \{\emptyset\}$ are two functions such that $\phi(I(+)M) = \psi(I)(+)N$ for some submodule N of M with $\psi(I)M \subseteq N$. Then the following conditions are equivalent:*

- (1) $I(+)M$ is a ϕ - n -absorbing primary ideal of $R(+)M$;
- (2) I is a ψ - n -absorbing primary ideal of R and if (a_1, \dots, a_{n+1}) is a ψ -($n+1$)-tuple, then the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ is in N for any elements $m_1, \dots, m_{n+1} \in M$.

Proof. (1) \Rightarrow (2) Assume that $I(+)M$ is a ϕ - n -absorbing primary ideal of $R(+)M$. Let $x_1 \cdots x_{n+1} \in I \setminus \psi(I)$ for some $x_1, \dots, x_{n+1} \in R$. Therefore

$$(x_1, 0) \cdots (x_{n+1}, 0) = (x_1 \cdots x_{n+1}, 0) \in I(+)M \setminus \phi(I(+)M),$$

because $\phi(I(+)M) = \psi(I)(+)N$. Hence either $(x_1, 0) \cdots (x_n, 0) = (x_1 \cdots x_n, 0) \in I(+)M$ or $(x_1, 0) \cdots \widehat{(x_i, 0)} \cdots (x_{n+1}, 0) = (x_1 \cdots \widehat{x_i} \cdots x_{n+1}, 0) \in \sqrt{I(+)M} = \sqrt{I}(+)M$ for some $1 \leq i \leq n$. So either $x_1 \cdots x_n \in I$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$ which shows that I is ψ - n -absorbing primary. For the second statement suppose that $a_1 \cdots a_{n+1} \in \psi(I)$, $a_1 \cdots a_n \notin I$ and $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \notin \sqrt{I}$ for all $1 \leq i \leq n$. If the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ is not in N , then

$$(a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \in I(+)M \setminus \psi(I)(+)N,$$

Thus either $(a_1, m_1) \cdots (a_n, m_n) \in I(+)M$ or $(a_1, m_1) \cdots \widehat{(a_i, m_i)} \cdots (a_{n+1}, m_{n+1}) \in \sqrt{I}(+)M$ for some $1 \leq i \leq n$. So either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, which is a contradiction.

(2) \Rightarrow (1) Suppose that $(a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \in I(+)M \setminus \psi(I)(+)N$ for some $a_1, \dots, a_{n+1} \in R$ and some $m_1, \dots, m_{n+1} \in M$. Clearly $a_1 \cdots a_{n+1} \in I$. If $a_1 \cdots a_{n+1} \in \psi(I)$, then the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ cannot be in N . Hence either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. If $a_1 \cdots a_{n+1} \notin \psi(I)$, then I ψ - n -absorbing primary implies that either $a_1 \cdots a_n \in I$ or $a_1 \cdots \widehat{a_i} \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Therefore we have either $(a_1, m_1) \cdots (a_n, m_n) \in I(+)M$ or $(a_1, m_1) \cdots \widehat{(a_i, m_i)} \cdots (a_{n+1}, m_{n+1}) \in \sqrt{I}(+)M$ for some $1 \leq i \leq n$. Consequently $I(+)M$ is a ϕ - n -absorbing primary ideal of $R(+)M$. \square

Corollary 4.2. *Let R be a ring, I be a proper ideal of R and M be an R -module. The following conditions are equivalent:*

- (1) $I(+)M$ is an n -absorbing primary ideal of $R(+)M$;
- (2) I is an n -absorbing primary ideal of R .

Proof. In Theorem 4.1 set $\phi = \emptyset$, $\psi = \emptyset$ and $N = M$. \square

Corollary 4.3. *Let R be a ring, I be a proper ideal of R and M be an R -module. The following conditions are equivalent:*

- (1) $I(+)M$ is a weakly n -absorbing primary ideal of $R(+)M$;
- (2) I is a weakly n -absorbing primary ideal of R and if (a_1, \dots, a_{n+1}) is an $(n+1)$ -tuple-zero, then the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ is zero for any elements $m_1, \dots, m_{n+1} \in M$.

Proof. In Theorem 4.1 set $\phi = 0$, $\psi = 0$ and $N = \{0\}$. \square

Corollary 4.4. *Let R be a ring, I be a proper ideal of R and M be an R -module. Then the following conditions are equivalent:*

- (1) $I(+)M$ is an n -almost n -absorbing primary ideal of $R(+)M$;
- (2) I is an n -almost n -absorbing primary ideal of R and if (a_1, \dots, a_{n+1}) is a ϕ_n -($n+1$)-tuple, then for any elements $m_1, \dots, m_{n+1} \in M$ the second component of $(a_1, m_1) \cdots (a_{n+1}, m_{n+1})$ is in $I^{n-1}M$.

Proof. Notice that $(I(+)M)^n = I^n(+)I^{n-1}M$. In Theorem 4.1 set $\phi = \phi_n$, $\psi = \phi_n$ and $N = I^{n-1}M$. \square

Corollary 4.5. *Let R be a ring, I be a proper ideal of R and M be an R -module such that $IM = M$. Then $I(+)M$ is an n -almost n -absorbing primary ideal of $R(+)M$ if and only if I is an n -almost n -absorbing primary ideal of R .*

Corollary 4.6. *Let R be a ring, I be a proper ideal of R and M be an R -module such that $IM = M$. Then $I(+)M$ is an ω - n -absorbing primary ideal of $R(+)M$ if and only if I is an ω - n -absorbing primary ideal of R .*

5. STRONGLY ϕ - n -ABSORBING PRIMARY IDEALS

Proposition 5.1. *Let I be a proper ideal of a ring R . Then the following conditions are equivalent:*

- (1) *I is strongly ϕ - n -absorbing primary;*
- (2) *For every ideals I_1, \dots, I_{n+1} of R such that $I \subseteq I_1$, $I_1 \cdots I_{n+1} \subseteq I \setminus \phi(I)$ implies that either $I_1 \cdots I_n \subseteq I$ or $I_1 \cdots \hat{I}_i \cdots I_{n+1} \subseteq \sqrt{I}$ for some $1 \leq i \leq n$.*

Proof. (1) \Rightarrow (2) is clear.

(2) \Rightarrow (1) Let J, I_2, \dots, I_{n+1} be ideals of R such that $J I_2 \cdots I_{n+1} \subseteq I$ and $J I_2 \cdots I_{n+1} \not\subseteq \phi(I)$. Then we have that $(J + I) I_2 \cdots I_{n+1} = (J I_2 \cdots I_{n+1}) + (I I_2 \cdots I_{n+1}) \subseteq I$ and $(J + I) I_2 \cdots I_{n+1} \not\subseteq \phi(I)$. Set $I_1 := J + I$. Then, by the hypothesis either $I_1 \cdots I_n \subseteq I$ or $I_2 \cdots I_{n+1} \subseteq \sqrt{I}$ or there exists $2 \leq i \leq n$ such that $(J + I) I_2 \cdots \hat{I}_i \cdots I_{n+1} \subseteq \sqrt{I}$. Therefore, either $J I_2 \cdots I_n \subseteq I$ or $I_2 \cdots I_{n+1} \subseteq \sqrt{I}$ or there exists $2 \leq i \leq n$ such that $J I_2 \cdots \hat{I}_i \cdots I_{n+1} \subseteq \sqrt{I}$. So I is strongly ϕ - n -absorbing primary. \square

Remark 5.2. Let R be a ring. Notice that $\text{Jac}(R)$ is a radical ideal of R . So $\text{Jac}(R)$ is a strongly n -absorbing ideal of R if and only if I is a strongly n -absorbing primary ideal of R .

Given any set X , one can define a topology on X where every subset of X is an open set. This topology is referred to as the discrete topology on X , and X is a discrete topological space if it is equipped with its discrete topology. We denote by $\text{Max}(R)$ the set of all maximal ideals of R .

Theorem 5.3. *Let R be a ring and $\text{Max}(R)$ be a discrete topological space. Then $\text{Max}(R)$ is an infinite set if and only if $\text{Jac}(R)$ is not strongly n -absorbing for every natural number n .*

Proof. (\Leftarrow) We can verify this implication without any assumption on $\text{Max}(R)$, by [2, Theorem 2.1].

(\Rightarrow) Notice that $\text{Max}(R)$ is a discrete topological space if and only if the Jacobson radical of R is the irredundant intersection of the maximal ideals of R , [1, Corollary 3.3]. Let $\text{Max}(R)$ be an infinite set. Assume that for some natural number n , $\text{Jac}(R)$ is a strongly n -absorbing ideal. Choose n distinct elements M_1, M_2, \dots, M_n of $\text{Max}(R)$. Set $\mathcal{M} := \{M_1, M_2, \dots, M_n\}$, and denote by \mathcal{M}^c the complement of \mathcal{M} in $\text{Max}(R)$. Since $\text{Jac}(R) = M_1 \cap M_2 \cap \cdots \cap M_n \cap (\bigcap_{M \in \mathcal{M}^c} M)$, then either $M_1 \cdots M_{i-1} M_{i+1} \cdots M_n (\bigcap_{M \in \mathcal{M}^c} M) \subseteq \text{Jac}(R)$ for some $1 \leq i \leq n$, or $M_1 M_2 \cdots M_n \subseteq \text{Jac}(R)$. In the first case we have $M_1 \cdots M_{i-1} M_{i+1} \cdots M_n (\bigcap_{M \in \mathcal{M}^c} M) \subseteq M_i$ and so $\bigcap_{M \in \mathcal{M}^c} M \subseteq M_i$, a contradiction. If $M_1 M_2 \cdots M_n \subseteq \text{Jac}(R)$, then $M_1 M_2 \cdots M_n \subseteq M$ for every $M \in \mathcal{M}^c$, and so again we reach a contradiction. Consequently $\text{Jac}(R)$ is not strongly n -absorbing. \square

In the next theorem we investigate ϕ - n -absorbing primary ideals over u -rings. Notice that any Bézout ring is a u -ring, [19, Corollary 1.2].

Theorem 5.4. *Let R be a u -ring and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then the following conditions are equivalent:*

- (1) *I is strongly ϕ - n -absorbing primary;*
- (2) *I is ϕ - n -absorbing primary;*
- (3) *For every elements $x_1, \dots, x_n \in R$ with $x_1 \cdots x_n \notin \sqrt{I}$ either*

$$(I :_R x_1 \cdots x_n) = (I :_R x_1 \cdots x_{n-1})$$

or

$$(I :_R x_1 \cdots x_n) \subseteq (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_n) \text{ for some } 1 \leq i \leq n-1 \text{ or}$$

$$(I :_R x_1 \cdots x_n) = (\phi(I) :_R x_1 \cdots x_n);$$
- (4) *For every t ideals I_1, \dots, I_t , $1 \leq t \leq n-1$, and for every elements $x_1, \dots, x_{n-t} \in R$ such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \not\subseteq \sqrt{I}$,*

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (I :_R x_1 \cdots x_{n-t-1} I_1 \cdots I_t)$$

or

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) \subseteq (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_{n-t} I_1 \cdots I_t)$$

for some $1 \leq i \leq n-t-1$ *or*

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots \hat{I}_j \cdots I_t)$$

for some $1 \leq j \leq t$ *or*

$$(I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (\phi(I) :_R x_1 \cdots x_{n-t} I_1 \cdots I_t).$$
- (5) *For every ideals I_1, I_2, \dots, I_n of R with $I_1 I_2 \cdots I_n \not\subseteq I$, either there is $1 \leq i \leq n$ such that $(I :_R I_1 \cdots I_n) \subseteq (\sqrt{I} :_R I_1 \cdots \hat{I}_i \cdots I_n)$ or $(I :_R I_1 \cdots I_n) = (\phi(I) :_R I_1 \cdots I_n)$.*

Proof. (1) \Rightarrow (2) It is clear.

(2) \Rightarrow (3) Suppose that $x_1, \dots, x_n \in R$ such that $x_1 \cdots x_n \notin \sqrt{I}$. By Theorem 2.3,

$$(I :_R x_1 \cdots x_n) \subseteq [\cup_{i=1}^{n-1} (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_n)] \cup (I :_R x_1 \cdots x_{n-1}) \cup (\phi(I) :_R x_1 \cdots x_n).$$

Since R is a u -ring we have either $(I :_R x_1 \cdots x_n) \subseteq (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_n)$ for some $1 \leq i \leq n-1$ or $(I :_R x_1 \cdots x_n) = (I :_R x_1 \cdots x_{n-1})$ or $(I :_R x_1 \cdots x_n) = (\phi(I) :_R x_1 \cdots x_n)$.

(3) \Rightarrow (4) We use induction on t . For $t = 1$, consider elements $x_1, \dots, x_{n-1} \in R$ and ideal I_1 of R such that $x_1 \cdots x_{n-1} I_1 \not\subseteq \sqrt{I}$. Let $a \in (I :_R x_1 \cdots x_{n-1} I_1)$. Then $I_1 \subseteq (I :_R a x_1 \cdots x_{n-1})$. If $a x_1 \cdots x_{n-1} \in \sqrt{I}$, then $a \in (\sqrt{I} :_R x_1 \cdots x_{n-1})$. If $a x_1 \cdots x_{n-1} \notin \sqrt{I}$, then by part (3), either $I_1 \subseteq (I :_R a x_1 \cdots x_{n-2})$ or $I_1 \subseteq (\sqrt{I} :_R a x_1 \cdots \hat{x}_i \cdots x_{n-1})$ for some $1 \leq i \leq n-2$ or $I_1 \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-1})$ or $I_1 \subseteq (\phi(I) :_R a x_1 \cdots x_{n-1})$. The first case implies that $a \in (I :_R x_1 \cdots x_{n-2} I_1)$. The second case implies that $a \in (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_{n-1} I_1)$ for some $1 \leq i \leq n-2$. The third case cannot be happen, because $x_1 \cdots x_{n-1} I_1 \not\subseteq \sqrt{I}$, and the last case implies that $a \in (\phi(I) :_R x_1 \cdots x_{n-1} I_1)$. Hence

$$(I :_R x_1 \cdots x_{n-1} I_1) \subseteq \cup_{i=1}^{n-2} (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_{n-1} I_1) \cup (\sqrt{I} :_R x_1 \cdots x_{n-1} I_1) \cup (I :_R x_1 \cdots x_{n-2} I_1) \cup (\phi(I) :_R x_1 \cdots x_{n-1} I_1).$$

Since R is a u -ring, then either $(I :_R x_1 \cdots x_{n-1} I_1) \subseteq (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_{n-1} I_1)$ for some $1 \leq i \leq n-2$, or $(I :_R x_1 \cdots x_{n-1} I_1) \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-1})$ or $(I :_R x_1 \cdots x_{n-1} I_1) = (I :_R x_1 \cdots x_{n-2} I_1)$ or $(I :_R x_1 \cdots x_{n-1} I_1) = (\phi(I) :_R x_1 \cdots x_{n-1} I_1)$. Now suppose $t > 1$ and assume that for integer $t-1$ the claim holds. Let x_1, \dots, x_{n-t} be elements of R and let I_1, \dots, I_t be ideals of R such that $x_1 \cdots x_{n-t} I_1 \cdots I_t \not\subseteq \sqrt{I}$. Consider element $a \in (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t)$. Thus $I_t \subseteq (I :_R a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1})$. If $a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1} \subseteq \sqrt{I}$, then $a \in (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots I_{t-1})$. If $a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1} \not\subseteq \sqrt{I}$, then by induction hypothesis, either

$$(I :_R a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots I_{t-1})$$

or

$$(I :_R a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) \subseteq (\sqrt{I} :_R a x_1 \cdots \hat{x}_i \cdots x_{n-t} I_1 \cdots I_{t-1})$$

for some $1 \leq i \leq n-t-1$ or

$$(I :_R a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) \subseteq (\sqrt{I} :_R a x_1 \cdots x_{n-t} I_1 \cdots \hat{I}_j \cdots I_{t-1})$$

for some $1 \leq j \leq t-1$ or

$$(I :_R a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) = (I :_R a x_1 \cdots x_{n-t-1} I_1 \cdots I_{t-1}),$$

or $(I :_R a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1}) = (\phi(I) :_R a x_1 \cdots x_{n-t} I_1 \cdots I_{t-1})$. Since $x_1 \cdots x_{n-t} I_1 \cdots I_t \not\subseteq \sqrt{I}$, then the first case cannot happen. Consequently, either $a \in (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_{n-t} I_1 \cdots I_t)$ for some $1 \leq i \leq n-t-1$ or $a \in (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots \hat{I}_j \cdots I_t)$ for some $1 \leq j \leq t-1$ or $a \in (I :_R x_1 \cdots x_{n-t-1} I_1 \cdots I_t)$, or $a \in (\phi(I) :_R x_1 \cdots x_{n-t} I_1 \cdots I_t)$. Hence

$$\begin{aligned} (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) &\subseteq [\cup_{i=1}^{n-t-1} (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_{n-t} I_1 \cdots I_t)] \\ &\cup [\cup_{j=1}^t (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots \hat{I}_j \cdots I_t)] \\ &\cup (I :_R x_1 \cdots x_{n-t-1} I_1 \cdots I_t) \\ &\cup (\phi(I) :_R x_1 \cdots x_{n-t} I_1 \cdots I_t). \end{aligned}$$

Now, since R is u -ring we are done.

(4) \Rightarrow (5) Let I_1, I_2, \dots, I_n be ideals of R such that $I_1 I_2 \cdots I_n \not\subseteq I$. Suppose that $a \in (I :_R I_1 I_2 \cdots I_n)$. Then $I_n \subseteq (I :_R a I_1 I_2 \cdots I_{n-1})$. If $a I_1 I_2 \cdots I_{n-1} \subseteq \sqrt{I}$, then $a \in (\sqrt{I} :_R I_1 I_2 \cdots I_{n-1})$. If $a I_1 I_2 \cdots I_{n-1} \not\subseteq \sqrt{I}$, then by part (4) we have either $I_n \subseteq (I :_R I_1 I_2 \cdots I_{n-1})$ or $I_n \subseteq (\sqrt{I} :_R a I_1 \cdots \hat{I}_i \cdots I_{n-1})$ for some $1 \leq i \leq n-1$ or $I_n \subseteq (\phi(I) :_R a I_1 I_2 \cdots I_{n-1})$. By hypothesis, the first case is not hold. The second case implies that $a \in (\sqrt{I} :_R I_1 \cdots \hat{I}_i \cdots I_n)$ for some $1 \leq i \leq n-1$. The third case implies that $a \in (\phi(I) :_R I_1 I_2 \cdots I_n)$. Similarly, since R is u -ring, there is $1 \leq i \leq n$ such that $(I :_R I_1 \cdots I_n) \subseteq (\sqrt{I} :_R I_1 \cdots \hat{I}_i \cdots I_n)$ or $(I :_R I_1 \cdots I_n) = (\phi(I) :_R I_1 \cdots I_n)$.

(5) \Rightarrow (1) This implication has an easy verification. \square

Remark 5.5. Note that in Theorem 5.4, for the case $n = 2$ and $\phi = \phi_\emptyset$ we can omit the condition u -ring, by the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them. So we conclude that an ideal I of a ring R is 2-absorbing primary if and only if it is strongly 2-absorbing primary.

Let R be a ring with identity. We recall that if $f = a_0 + a_1X + \cdots + a_tX^t$ is a polynomial on the ring R , then *content* of f is defined as the R -ideal, generated by the coefficients of f , i.e. $c(f) = (a_0, a_1, \dots, a_t)$. Let T be an R -algebra and c the function from T to the ideals of R defined by $c(f) = \cap\{I \mid I \text{ is an ideal of } R \text{ and } f \in IT\}$ known as the content of f . Note that the content function c is nothing but the generalization of the content of a polynomial $f \in R[X]$. The R -algebra T is called a *content R -algebra* if the following conditions hold:

- (1) For all $f \in T$, $f \in c(f)T$.
- (2) (Faithful flatness) For any $r \in R$ and $f \in T$, the equation $c(rf) = rc(f)$ holds and $c(1_T) = R$.
- (3) (Dedekind-Mertens content formula) For each $f, g \in T$, there exists a natural number n such that $c(f)^nc(g) = c(f)^{n-1}c(fg)$.

For more information on content algebras and their examples we refer to [17], [18] and [20]. In [16] Nasehpour gave the definition of a Gaussian R -algebra as follows: Let T be an R -algebra such that $f \in c(f)T$ for all $f \in T$. T is said to be a Gaussian R -algebra if $c(fg) = c(f)c(g)$, for all $f, g \in T$.

Example 5.6. ([16]) Let T be a content R -algebra such that R is a Prüfer domain. Since every nonzero finitely generated ideal of R is a cancellation ideal of R , the Dedekind-Mertens content formula causes T to be a Gaussian R -algebra.

In the following theorem we use the functions ϕ_R and ϕ_T that defined just prior to Theorem 2.25.

Theorem 5.7. *Let R be a Prüfer domain, T a content R -algebra and I an ideal of R . Then I is a ϕ_R - n -absorbing primary ideal of R if and only if IT is a ϕ_T - n -absorbing primary ideal of T .*

Proof. Assume that I is a ϕ_R - n -absorbing primary ideal of R . Let $f_1f_2 \cdots f_{n+1} \in IT \setminus \phi_T(IT)$ for some $f_1, f_2, \dots, f_{n+1} \in T$ such that $f_1f_2 \cdots f_n \notin IT$. Then $c(f_1f_2 \cdots f_{n+1}) \subseteq I$. Since R is a Prüfer domain and T is a content R -algebra, then T is a Gaussian R -algebra. Therefore $c(f_1f_2 \cdots f_{n+1}) = c(f_1)c(f_2) \cdots c(f_{n+1}) \subseteq I$. If $c(f_1f_2 \cdots f_{n+1}) \subseteq \phi_R(I) = \phi_T(IT) \cap R$, then

$$f_1f_2 \cdots f_{n+1} \in c(f_1f_2 \cdots f_{n+1})T \subseteq (\phi_T(IT) \cap R)T \subseteq \phi_T(IT),$$

which is a contradiction. Hence $c(f_1)c(f_2) \cdots c(f_{n+1}) \subseteq I$ and $c(f_1)c(f_2) \cdots c(f_{n+1}) \not\subseteq \phi_R(I)$. Since R is a u -domain, I is a strongly ϕ_R - n -absorbing primary ideal of R , by Theorem 5.4, and this implies either $c(f_1)c(f_2) \cdots c(f_n) \subseteq I$ or $c(f_1) \cdots \widehat{c(f_i)} \cdots c(f_{n+1}) \subseteq \sqrt{I}$ for some $1 \leq i \leq n$. In the first case we have $f_1f_2 \cdots f_n \in c(f_1f_2 \cdots f_n)T \subseteq IT$, which contradicts our hypothesis. In the second case we have $f_1 \cdots \widehat{f_i} \cdots f_{n+1} \in (\sqrt{I})T \subseteq \sqrt{IT}$ for some $1 \leq i \leq n$. Consequently IT is a ϕ_T - n -absorbing primary ideal of T .

For the converse, note that since T is a content R -algebra, $IT \cap R = I$ for every ideal I of R . Now, apply Theorem 2.25. \square

The algebra of all polynomials over an arbitrary ring with an arbitrary number of indeterminates is an example of content algebras.

Corollary 5.8. *Let R be a Prüfer domain and I be an ideal of R . Then I is a ϕ_R - n -absorbing primary ideal of R if and only if $I[X]$ is a $\phi_{R[X]}$ - n -absorbing primary ideal of $R[X]$.*

As two special cases of Corollary 5.8, when $\phi_R = \phi_T = \emptyset$ and $\phi_R = \phi_T = 0$ we have the following result.

Corollary 5.9. *Let R be a Prüfer domain and I be an ideal of R . Then I is an n -absorbing primary ideal of R if and only if $I[X]$ is an n -absorbing primary ideal of $R[X]$.*

REFERENCES

- [1] P. Alirio J. Peña; Jorge E. Vielma, *Isolated points and redundancy*, Comment. Math. Univ. Carolin., **52** (2011), 145–152.
- [2] D. F. Anderson and A. Badawi, *On n -absorbing ideals of commutative rings*, Comm. Algebra, **39** (2011), 1646–1672.
- [3] D. D. Anderson and M. Batanieh, *Generalizations of prime ideals*, Comm. Algebra, **36** (2008), 686–696.
- [4] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math., **29** (2003), 831–840.
- [5] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., **75** (2007), 417–429.
- [6] A. Badawi, E. Houston, *Powerful ideals, strongly primary ideals, almost pseudo-valuation domains and conductive domains*, Comm. Algebra, **30** (2002), 1591–1606.
- [7] A. Badawi, Ü. Tekir and E. Yetkin, *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc., **51** (4) (2014), 1163–1173.
- [8] A. Badawi and A. Yousefian Darani, *On weakly 2-absorbing ideals of commutative rings*, Houston J. Math., **39** (2013), 441–452.
- [9] S. M. Bhatwadekar, P. K. Sharma, *Unique factorization and birth of almost primes*, Comm. Algebra, **33** (2005), 43–49.
- [10] S. Ebrahimi Atani, F. Farzalipour, *On weakly primary ideals*, Georgian Math. J., **12** (2005), 423–429.
- [11] M. Ebrahimpour, R. Nekooei, *On generalizations of prime ideals*, Comm. Algebra, **40** (2012), 1268–1279.
- [12] D. J. Fieldhouse, Purity And Flatness, *McGill University, Montreal, Canada*, 1967.
- [13] R. Gilmer, Multiplicative ideal theory. *Queens Papers Pure Appl. Math. 90, Queens University, Kingston*, 1992.
- [14] J. Hukaba, Commutative rings with zero divisors, *Marcel Dekker, Inc., New York*, 1988.
- [15] H. Mostafanasab, A. Yousefian Darani, *On weakly n -absorbing ideals of commutative rings*, submitted.
- [16] P. Nasehpour, *On the Anderson-Badawi $\omega_{R[X]}(I[X]) = \omega_R(I)$ conjecture*, arXiv:1401.0459, (2014).
- [17] D. G. Northcott, *A generalization of a theorem on the content of polynomials*, Proc. Cambridge Phil. Soc., **55** (1959), 282–288.
- [18] J. Ohm and D. E. Rush, *Content modules and algebras*, Math. Scand., **31** (1972), 49–68.
- [19] P. Quartararo and H. S. Butts, *Finite unions of ideals and modules*, Proc. Amer. Math. Soc., **52** (1975), 91–96.
- [20] D. E. Rush, *Content algebras*, Canad. Math. Bull., **21** (3) (1978), 329–334.
- [21] R.Y. Sharp, Steps in commutative algebra, *Second edition, Cambridge University Press, Cambridge*, 2000.
- [22] A. Yousefian Darani, *Generalizations of primary ideals in commutative rings*, Novi Sad J. Math., **42**(1) (2012), 27–35.

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